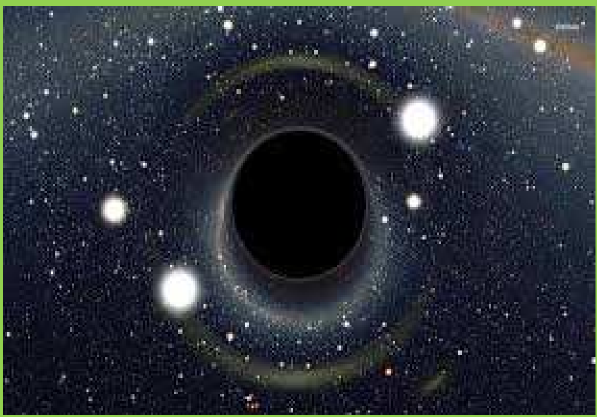
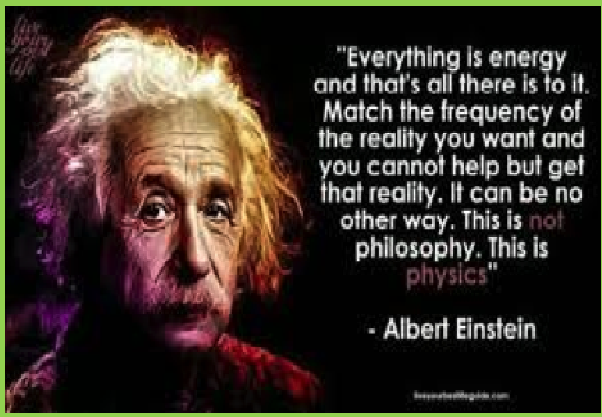
BSCPH- 101

B. Sc. I YEAR

MECHANICS





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BSCPH-101

Mechanics

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Contents

Course 1: Mechanics

Course code: BSCPH101 **Credit: 3**

UNIT 1: VECTOR

STRUCTURE:

1.0 Objective:

After reading this unit you will be able to understand:
* Defining vector.
*Vector representation, addition, subtraction
*Orthogonal representation
*Multiplication of vectors
*Scalar product, vector product
*Scalar triple product and vector triple product

1.1 Introduction:

On the basis of direction, the physical quantities may be divided into two main classes.

1.1.1 Scalar quantities: The physical quantities which do not require direction for their representation. These quantities require only magnitude and unit and are added according to the usual rules of algebra. Examples of these quantities are: mass, length, area, volume, distance, time speed, density, electric current, temperature, work etc.

1.1.2 Vector quantities: The physical quantities which require both magnitude and direction and which can be added according to the vector laws of addition are called vector quantities or vector. These quantities require magnitude, unit and direction. Examples are weight, displacement, velocity, acceleration, magnetic field, current density, electric field, momentum angular velocity, force etc.

1.2 Vector representation:

Any vector quantity say A, is represented by putting a small arrow above the physical quantity like $\vec{A}$. In case of print text a vector quantity is represented by bold type letter like **A**. The vector can be represented by both capital and small letters. The magnitude of a vector quantity A is denoted by $|\vec{A}|$ or *mod A* or some time light forced italic letter *A*. We should understand following types of vectors and their representations.

1.2.1 Unit vector

A unit vector of any vector quantity is that vector which has unit magnitude. Suppose $\vec{A}\ is\ a\ vector$ then unit vector is defined as

$$\hat{A} = \frac{\vec{A}}{|A|}$$

The unit vector is denoted by $\hat{A}$ and read as 'A *unit vector or A hat'.* It is clear that the magnitude of unit vector is always 1. A unit vector merely indicates direction only. In Cartesian coordinate system, the unit vector along x, y and x axis are represented by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ respectively as shown in figure 1.1.

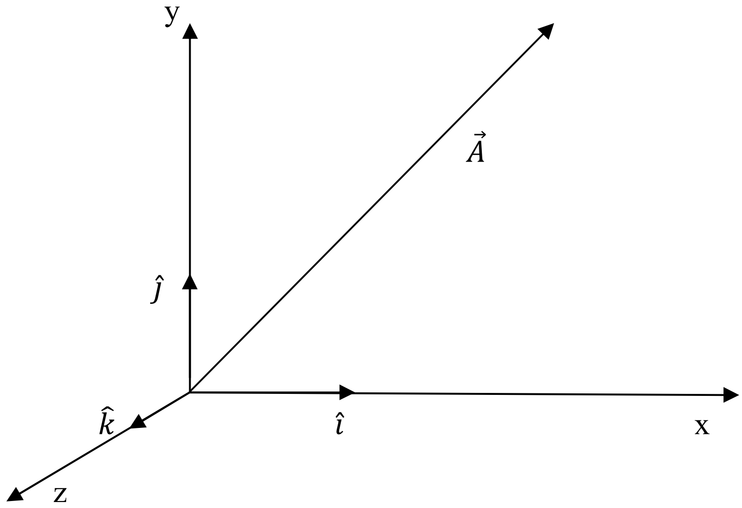


Figure 1.1

Any vector in Cartesian coordinate system can be represented as

$$A = \hat{\imath}\,\mathrm{A_x} + \hat{\jmath}\;\mathrm{A_y} + \hat{k}\,\mathrm{A_z}$$

Where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are unit vector along x, y z axis and, $\mathrm{A_x}$, $\mathrm{A_y}$, $\mathrm{A_z}$ are the magnitudes projections or components of $\vec{A}$ along x, y, z axis respectively.

The unit vector in Cartesian coordinate system can be given as:

$$\hat{A} = \frac{\hat{\imath}\,\mathrm{A_x} + \hat{\jmath}\;\mathrm{A_y} + \hat{k}\,\mathrm{A_z}}{\sqrt{\mathrm{A_x}^{2} + \mathrm{B_x}^{2} + \mathrm{C_x}^{2}}}$$

1.2.2. Zero vector or Null vector:

A vector with zero magnitude is called zero vector or null vector. The condition for null vector is $|\vec{A}| = 0$

1.2.3 Equal vectors:

If two vectors have same magnitude and same direction, the vectors are called equal vector.

1.2.4 Like vectors:

If two or more vectors have same direction, but may have different magnitude, then the vectors are called like vectors.

1.2.5 Negative vector:

A vector is called negative vector with reference to another one, if both have same magnitude but opposite directions.

1.2.6 Collinear vectors:

All the vectors parallel to each other are called collinear vectors. Basically collinear means the line of action is along the same line.

1.2.6 Coplanar vector:

All the vectors whose line of action lies on a same plane are called coplanar vectors. Basically coplanar means lies on the same plane.

1.2.7. Graphical representation of vectors:

Graphically a vector quantity is represented by an arrow shaped straight line, with suitable length which represents magnitude, and the direction of arrow represents direction of vector quantity. For example, if a force $\vec{A}$ is directed towards east and another force $\vec{B}$ is directed toward north-west then these forces can be represented as shown in figure 1.2.

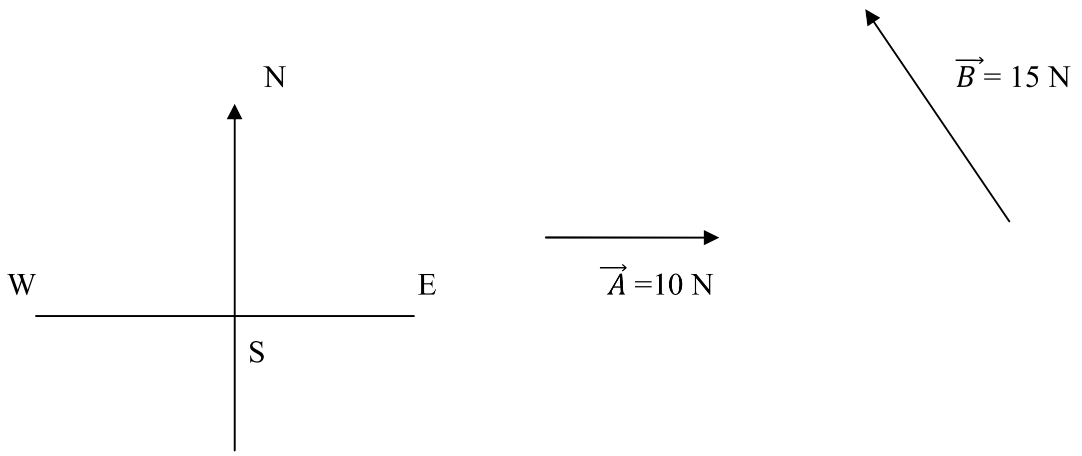


Figure 1.2

1.2.8 Addition and subtraction of vectors:

The addition of two vectors can be performed by following two laws.

(A) The parallelogram law:

According to this law, if two vectors $\vec{A}$ and $\vec{B}$ are represented by two adjacent sides of a parallelogram as show in figure 1.3, then the sum of these two vectors or resultant $\vec{R}$ is represented by the diagonal of Parallelogram.

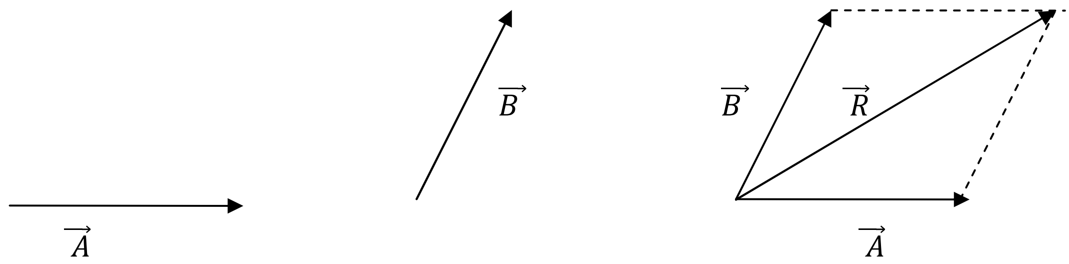


Figure 1.3

If vector $\vec{A}$ and $\vec{B}$ are represented by the sides of a parallelogram as shown in figure 1.4 and the angle between $\vec{A}$ and $\vec{B}$ is θ, and resultant $\vec{R}$ makes angle α with vector $\vec{A}$ then magnitude of $\vec{R}$ is

$$|R| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

The angle α is given as

$$\alpha = tan^{-1}\frac{B\sin\theta}{A + Bcos\,\theta}$$

You should notice that all three vectors $\vec{A}$, $\vec{B}$ and $\vec{R}$ are concurrent *i.e.* vectors acting on the same point O.

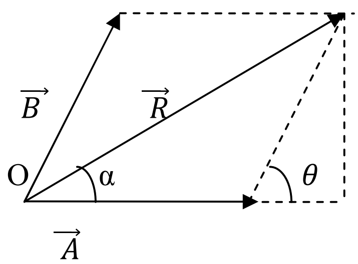


Figure 1.4

(B) Triangle law:

According to this law if a vector is placed at the head of another vector, and these two vectors represent two sides of a triangle then the third side or a vector drawn for the tail end of first to the head end of second represents the resultant of these two vectors. If vectors $\vec{A}$ and $\vec{B}$ are two vector as shown in figure 1.5, then resultant $\vec{R}$ can be obtained by applying triangle law.

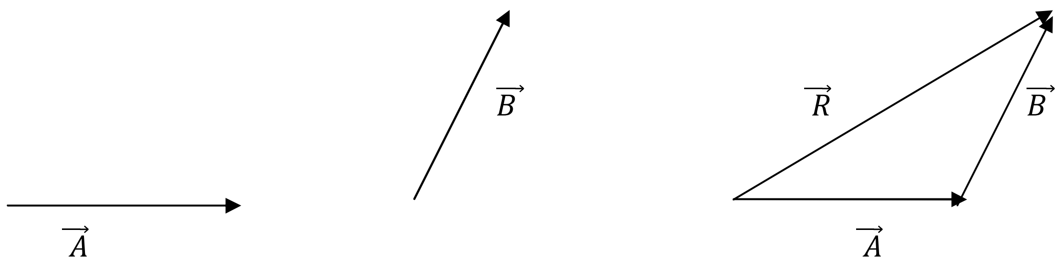


Figure 1.5

(c) Polygon law of vector addition:

This law is used for the addition of more than two vectors. According to this law if we have a large number of vectors, place the tail end of each successive vector at the head end of previous one. The resultant of all vectors can be obtained by drawing a vector from the tail end of first to the head end of the last vector. Figure 1.6 shows the polynomial law of vector addition different vectors $\vec{A}$,$\vec{B}$, $\vec{C}$, $\vec{D}$, $\vec{E}$ etc. and $\vec{R}$ is resultant vector.

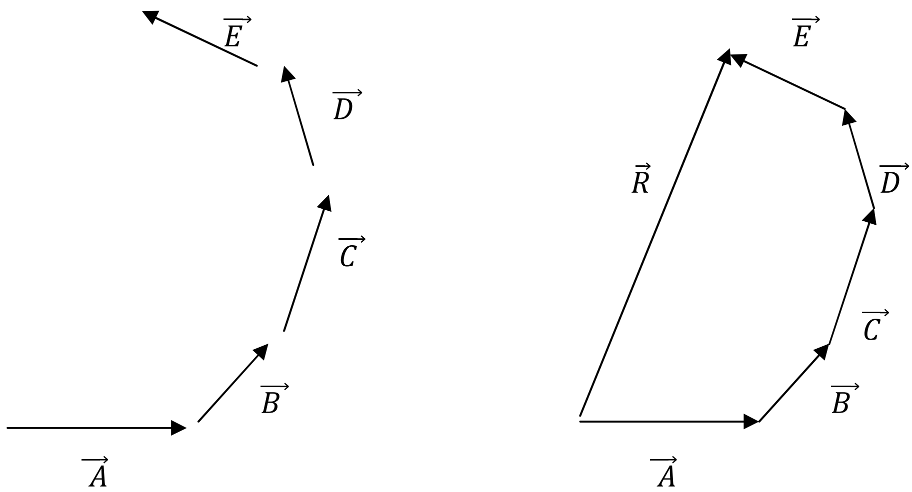


Figure 1.6

1.2.9 Resolution of vector:

A vector can be resolved into two or more vectors and these vectors can be added in accordance with the polygon law of vector addition, and finally original vector can be obtained. If a vector is resolved into three components which are mutually perpendicular to each other then these are called rectangular components or mutual perpendicular components of a vector. These components are along the three coordinate axes x, y and z respectively as show in figure 1.7.

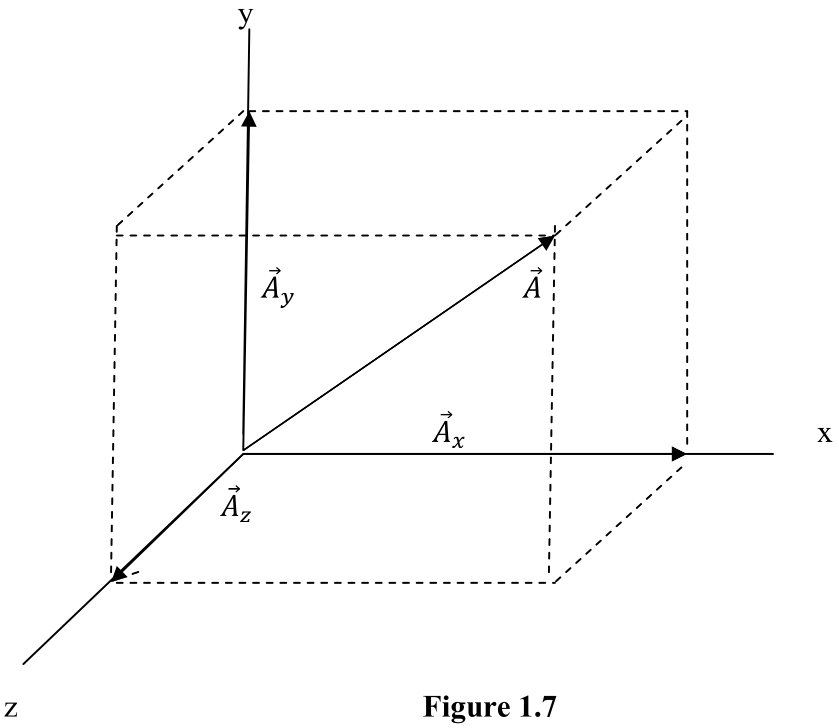


Figure 1.7

If the unit vectors along x, y and x axis are represented by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ respectively then any vector $\vec{A}$ can be give as

$$\vec{A} = \hat{\imath}\,\mathrm{A_x} + \hat{\jmath}\,\mathrm{A_y} + \hat{k}\,\mathrm{A_z}$$

$\vec{A}$ constitutes the diagonal of a parallelepiped, and $\mathrm{A_x}$, $\mathrm{A_y}$ and $\mathrm{A_z}$ are the edges along x, y and z axes respectively. $\vec{A}$ is polynomial addition of vectors $\mathrm{A_x}$, $\mathrm{A_y}$ and $\mathrm{A_z}$. The rectangular components $\mathrm{A_x}$, $\mathrm{A_y}$ and $\mathrm{A_z}$ can be considered as orthogonal projections of vector $\vec{A}$ on x, y and z axis respectively. Mathematically, the magnitude of vector $\vec{A}$ can be given as:

$$\boldsymbol{A} = |\vec{A}| = \sqrt{A_x^{\ 2} + A_y^{\ 2} + A_z^{\ 2}}$$

1.2.10 Direction cosines:

The cosine of angles which the vector $\vec{A}$ makes with three mutual perpendicular axes x, y and z are called direction cosine and generally represented by l, m, n respectively. In figure 1.8 vector $\vec{A}$ makes angle α, β *and* γ with axis x, y and z respectively. Then

$$\cos\alpha = \frac{A_x}{A} = \frac{A_x}{\sqrt{A_x^{\ 2} + A_y^{\ 2} + A_z^{\ 2}}}\ ; \ \cos\beta = \frac{A_y}{A} = \frac{A_y}{\sqrt{A_x^{\ 2} + A_y^{\ 2} + A_z^{\ 2}}}\ ;$$

$$\cos\gamma = \frac{A_z}{A} = \frac{A_z}{\sqrt{A_x^{\ 2} + A_y^{\ 2} + A_z^{\ 2}}}$$

Where A_x , A_y and A_z are the projection or intercepts of vector $\vec{A}$ along x, y and z axes respectively. The $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines.

$l = \cos\alpha$; $m = \cos\beta$; $n = \cos\gamma$

Mathematically

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ or $l^2 + m^2 + n^2 = 1$

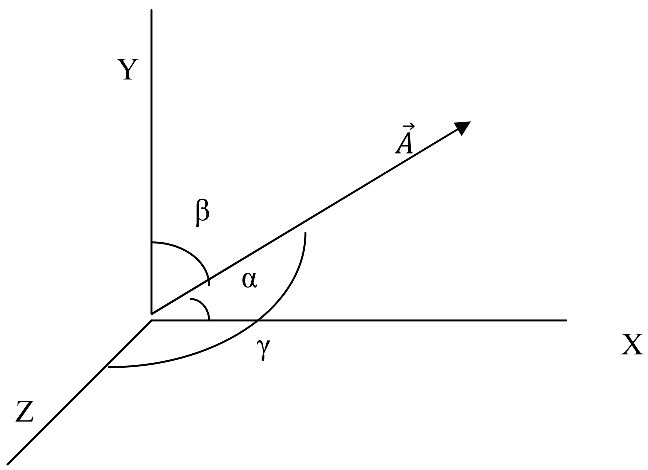


Figure 1.8

1.2.11 Position vector:

In Cartesian co-ordinate system the position of any point P(x, y, z) can be represented by a vector **r**, with respect to origin O then the vector **r** is called position vector of point P. Position vector is often denoted by $\bar{r}$. Figure1.9 shows the position vector of a point P in Cartesian coordinate system. If we have two vectors $\vec{P}$ and $\vec{Q}$ with position vectors $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ respectively then

$$\boldsymbol{r_1} = \hat{\imath}\, x_1 + \hat{\jmath}\ y_1 + \hat{k}\, z_1$$

$$\boldsymbol{r_2} = \hat{\imath}\, x_2 + \hat{\jmath}\ y_2 + \hat{k}\, z_2$$

Where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of point P and Q respectively.

Now the vector PQ can be given as

PQ = OQ – OP (∴ OP + PQ = OQ)

$$\bar{r} = \bar{r_2} - \bar{r_1}$$

Therefore, vector PQ = position vector of Q − position vector of P

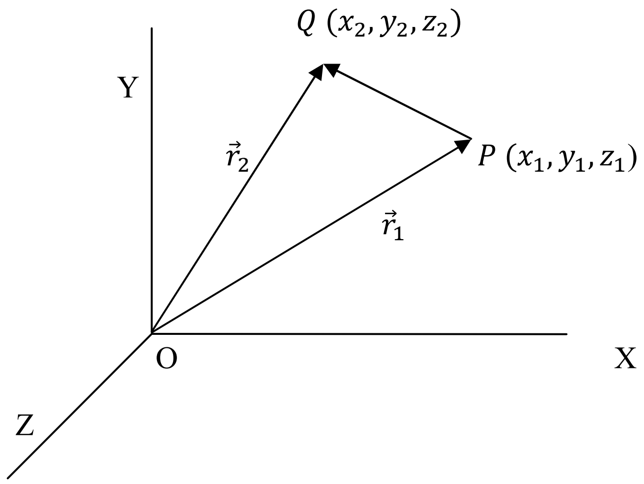


Figure1.9

1.3 Multiplication of vectors:

1.3.1 Multiplication and division of a vector by scalar:

If a vector **P** is multiplied by a scalar quantity m then its magnitude becomes m times. For example if m is a scalar and $\vec{A}$ is a vector then its magnitude becomes m times. Similarly, in case of division of a vector A by a non zero scalar quantity n, its magnitude becomes 1/n times.

1.3.2 Product of two vectors:

There are two distinct ways in which we can define the product of two vectors.

1.3.2.1 Scalar product or dot product:

Scalar product of two vectors **P** and **Q** is defined as the product of magnitude of two vectors P and Q and cosine of the angle between the directions of these vectors.

If θ is the angle between two vectors $\vec{P}$ and $\vec{Q}$, then dot product (*read as* $\vec{P}\ dot\ \vec{Q}$) of two vectors is given by-

$$\overrightarrow{P}.\ \vec{Q}\ = PQ\cos\theta\ = P\ (Q\cos\theta)$$

$$= P\ (projection\ of\ vector\ Q\ on\ P) =\ P.MN$$

The figure 1.10 shows the dot product. The resultant of dot product or scalar product of two vectors is always a scalar quantity. In physics the dot product is frequently used, the simplest example is work which is dot product of force and displacement vectors.

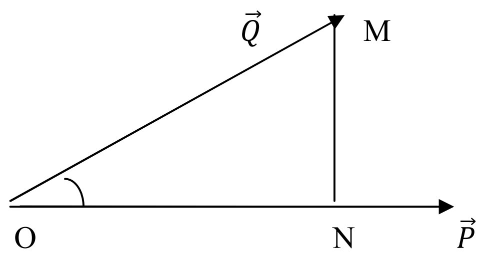


Figure1.10

Important properties of dot product

(i) Condition for two collinear vectors:

If two vectors are parallel or angle between two vectors is 0 or π, then vectors are called collinear. In this case

$$\overrightarrow{P}.\ \vec{Q}\ = PQcos\ 0^o\ = PQ$$

Then the product of two vectors is same as the product of their magnitudes.

(ii) Condition for two vector to be perpendicular to each other:
If two vectors are perpendicular to each other then the angle between these two vectors is 90^0, then

$$\vec{P}.\vec{Q} = PQ\cos\,90^o = 0$$

Hence two vectors are perpendicular to each other if and only if their dot product is zero.

In case of unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ we know that these vectors are perpendicular to each other then

$\hat{\imath}\,.\,\hat{\jmath} = \hat{\imath}.\,\hat{k} = \hat{k}.\hat{\imath} = 0$

similarly

$\hat{\imath}\,.\,\hat{\imath} = \hat{\jmath}.\,\hat{\jmath} = \hat{k}.\,\hat{k} = 1$

(iii) Commutative law holds:
In case of vector dot product the commutative law holds. Then

$$\vec{P}.\vec{Q} = \vec{Q}.\vec{P}$$

(iv) Distributive property of scalar product:
If P, Q and R are three vectors then according to distributive law

$\vec{P}.(\vec{Q}+\vec{R}) = \vec{P}.\vec{Q}+\vec{P}.\vec{R}$

Example 1.1 Show that vector $\vec{A} = 3i + 6j - 2k$ and $\vec{B} = 4i - \jmath + 3k$ are mutually perpendicular.

Solution: If the angle between $\vec{A}$ and $\vec{B}$ is θ then

$\vec{A}\,.\,\vec{B} = A\,B\cos\theta$

$\cos\theta = \frac{\vec{A}.\vec{B}}{AB} = \frac{(3i+6j-2k).(4i-j+3k)}{\sqrt{(A_x^2+A_y^2+A_z^2)}\sqrt{(B_x^2+B_y^2+B_z^2}} = 0$

$\cos\theta = 0, \theta = 90^0$

Therefore the vectors are mutually perpendicular.

Example 1.2 A particle moves from a point (3,-4,-2) meter to another point (5,-6, 2) meter under the influence of a force $\vec{F} = (-3\hat{\imath} + 4\hat{\jmath} + 4\hat{k}\,)$ N. Calculate the work done by the force.
Solution: Suppose the particle moves from point A to B. Then displacement of particle is given by

$$\vec{r} = position\ vetor\ of\ B - position\ vetor\ A$$
$$\vec{r} = [(5-3)i + (-6+4)j + (2+2)k] \text{ meter}$$
$$\vec{r} = (2i - 2j + 4k) \text{ meter}$$

Work done = $\vec{F}.\vec{r}$ = $[(-3\hat{\imath} + 4\hat{\jmath} + 4\hat{k}\ ).(\ 2i - 2j + 4k)$ N-meter =2 joule.

1.3.2.2 Vector product or Cross Product

The vector product or cross product of two vectors is a vector quantity and defined as a vector whose magnitude is equal to the product of magnitudes of two vectors and sine of angle between them.

If $\vec{A}$ and $\vec{B}$ are two vectors then cross product of these two vectors is denoted by $\vec{A} \times \vec{B}$ (*read as* $\vec{A}\ cross\ \vec{B}$) and given as

$$\vec{A} \times \vec{B} = ABsin\emptyset\ \hat{n} = \vec{C}$$

Where $\emptyset$ is the angle between vectors $\vec{A}$ and $\vec{B}$, and $\hat{n}$ is the unit vector perpendicular to both $\vec{A}$ and $\vec{B}\left(i.e. normal\ to\ the\ plane\ containg\ \vec{A}\ and\ \vec{B}\right)$.

Suppose $\vec{A}$ is along x axis and $\vec{B}$ is along y axis then vector product can be considered as an area of parallelogram OPQR as shown is figure 1.11 in XY plane whose sides are $\vec{A}$ and $\vec{B}$ and direction is perpendicular to plane OPQR i.e. along z axis. The cross product $\vec{A}$ and $\vec{B}$ is positive if direction of $\emptyset$ ($\vec{A}\ to\ \vec{B}$) is positive or rotation is anticlockwise as show in figure 1.11, and negative if the rotation of $\emptyset$ ($\vec{A}\ to\ \vec{\boldsymbol{B}}$) is clockwise (figure 1.12).

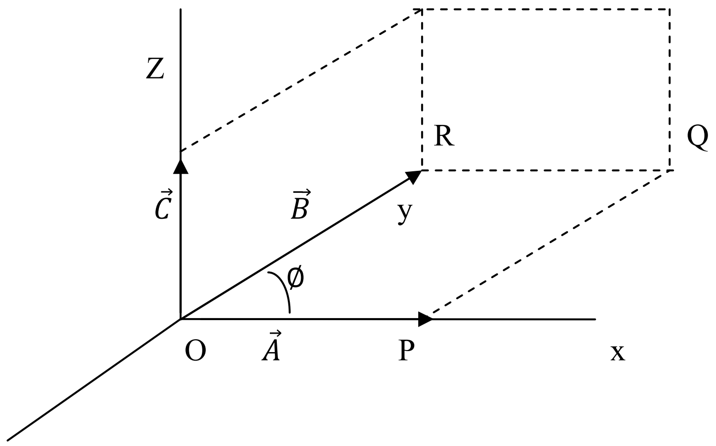


Figure1.11

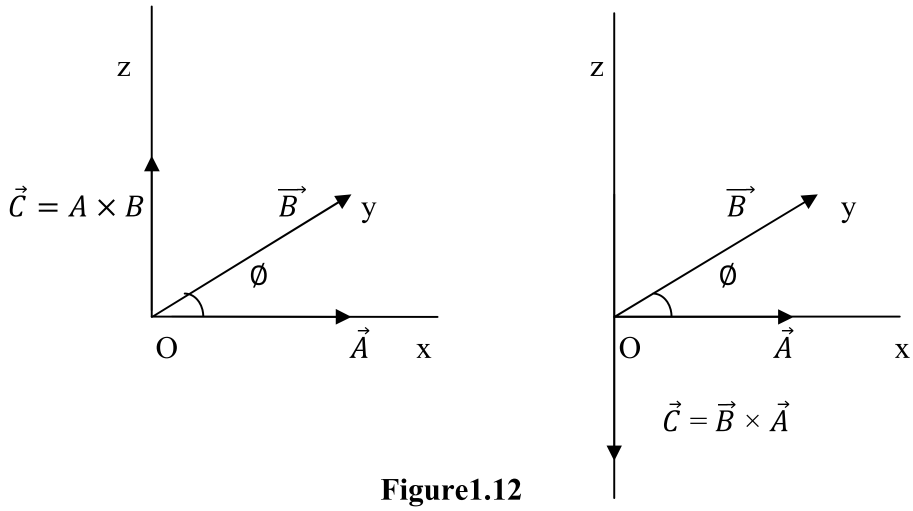


Figure1.12

Important properties of vector product

(i) Commutative law does not hold: From the definition of vector product of two vectors $\vec{A}$ and $\vec{B}$ the vector products are defined as

$\vec{A} \times \vec{B}$= AB sinØ $\hat{n}$

$\vec{B} \times \vec{A}$ = AB sinØ $(-\hat{n})$ = $-$AB sinØ $\hat{n}$ = $-\vec{A} \times \vec{B}$

Since in case of $\vec{B} \times \vec{A}$ the angle of rotation becomes opposite to case $\vec{A} \times \vec{B}$, hence product becomes negative.

Therefore, $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

(ii) Distributive law holds:
In case of vector product the distribution law holds.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(iii) Product of equal vectors
If two vectors are equal then the angle between them is zero, and vector product becomes

$$\vec{A} \times \vec{A} = |A||A| sin\emptyset\, \hat{n} = 0$$

Hence the vector product of two equal vectors in always zero.

In case of Cartesian coordinate system if $\hat{\imath}, \hat{\jmath}, \hat{k}$are unit vectors along x, y and z axes then

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$$

(iv) Collinear vectors: Collinear vectors are vectors parallel to each other. The angles between collinear vectors are always zero therefore

$$\vec{A} \times \vec{B} = |A||B|sin\emptyset\,\hat{n} = 0$$

Thus two vectors are parallel or anti-parallel or collinear if its vector product is 0.

(v) Vector product of orthogonal vector : If two vectors $\vec{A}$ and $\vec{B}$ are orthogonal to each other then angle between such vectors is $\emptyset = 90°$ therefore

$$\vec{A} \times \vec{B} = AB\ sin\emptyset\,\hat{n}$$
$$\vec{A} \times \vec{B} = |A||B|\,\hat{n}$$

In Cartesian coordinate system if $\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vector along x, y and z axes then

$$\boldsymbol{\hat{\imath} \times \hat{\jmath} = \hat{k} \quad \hat{\jmath} \times \hat{k} = \hat{\imath}\ and\ \ \hat{k} \times \hat{\imath} = \hat{\jmath}}$$
$$\hat{\jmath} \times \hat{\imath} = -\hat{k} \quad \hat{k} \times \hat{\jmath} = -\hat{\imath}\ and\ \ \hat{\imath} \times \hat{k} = \hat{\jmath}$$

(vi) Determinant form of vector product: If $\vec{A}$and $\vec{B}$ are two vectors given as

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

Then

$$\vec{A} \times \vec{B} = \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$
$$= A_xB_x\hat{\imath} \times \hat{\imath} + A_xB_y\hat{\imath} \times \hat{\jmath} + A_xB_z\hat{\imath} \times \hat{k} + A_yB_x\hat{\jmath} \times \hat{\imath} + A_yB_y\hat{\jmath} \times \hat{\jmath} +$$
$$+A_yB_z\hat{\jmath} \times \hat{k} + A_zB_x\hat{k} \times \hat{\imath} + A_zB_y\hat{k} \times \hat{\jmath} + A_zB_z\hat{k} \times \hat{k}$$

$$= A_xB_y\hat{k} - A_xB_z\hat{\jmath} - A_yB_x\hat{k} + A_yB_x\hat{\imath} + A_zB_x\hat{\jmath} - A_zB_y\hat{\imath}$$

(Since $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0\ and\ \hat{\imath} \times \hat{k} = -\hat{\jmath}, \hat{\jmath} \times \hat{\imath} = -\hat{k}, \hat{k} \times \hat{\jmath} = -\hat{\imath}$)

$$= \hat{\imath}(A_yB_z - A_zB_y) - \hat{\jmath}\,(A_xB_z - A_zB_x) +\ k(A_xB_y - A_yB_x)$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Physical significance of vector product:

In physics, numbers of physical quantities are defined in terms of vector products. Some basic examples are illustrated below.

(i) **Torque:** Torque or moment of force is define as

$$\vec{\tau} = \vec{r} \times \vec{f}$$

Where $\vec{\tau}$ is torque, $\vec{r}$ is position vector of a point P where the force $\vec{f}$is applied. (Figure 1.13)

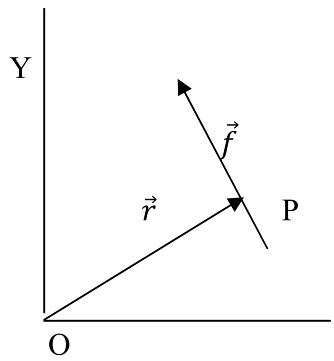


Figure 1.13

(ii) **Lorentz force on a moving charge in magnetic field**: if a charge q is moving in a magnetic field $\vec{B}$with a velocity $\vec{V}$at an angle with the direction of magnetic field then force $\vec{F}$experienced by the charged particle is give as;

$$\vec{F} = q(\vec{V} \times \vec{B})$$

This force is called Lorentz force and its direction is perpendicular to the direction of both velocity and magnetic field B.

(iii) **Angular Momentum**: Angular momentum is define as the moment of the momentum and given as:

$$\vec{L} = \vec{r} \times \vec{p}$$

Where $\vec{r}$ is the radial vector of circular motion and $\vec{p}$ is the linear moment of the body under circular motion, and $\vec{L}$ is angular momentum along the direction perpendicular to both $\vec{r}$ and $\vec{p}$. The law of conservation of angular momentum is a significant property in all circular motions.

1.3.3. Product of three vectors:

If we consider three vectors$\vec{A}$,$\vec{B}$$and$ $\vec{C}$, we can define two types of triple products known as scalar triple product and vector triple product.

1.3.3.1 Scalar Triple product:

Let us consider three vectors $\vec{A}$, $\vec{B}$and $\vec{C}$ then the scalar triple product of these three vectors is defined as $\vec{A}$.($\vec{B} \times \vec{C}$) and denoted as $[\vec{A}\vec{B}\vec{C}]$. This is a scalar quantity. If we consider $\overrightarrow{A}$, $\vec{B}$ and $\vec{C}$ the three sides of a parallelopiped as shown in figure 1.14 then $\vec{B} \times \vec{C}$ is a vector which represents the area of parallelogram OBDC which is the base of the parallelogram. The direction of $\vec{B} \times \vec{C}$ is naturally along Z axis (perpendicular to both $\vec{B}$ and $\vec{C}$). If $\emptyset$ is the angle between the direction of vectors ($\vec{B} \times \vec{C}$) and vector $\vec{A}$, then the dot product of vectors ($\vec{B} \times \vec{C}$) and vector $\vec{A}$ is given as (figure 1.14)

$$\vec{A} \,.\, (\vec{B} \times \vec{C}) = |A||\vec{B} \times \vec{C}|cos\emptyset = ACos\emptyset(\vec{B} \times \vec{C}) = h.(\vec{B} \times \vec{C})$$

= Vertical height of parallelogram × area of base of parallelogram

= Volume of parallelogram = $[A\ B\ C]$.

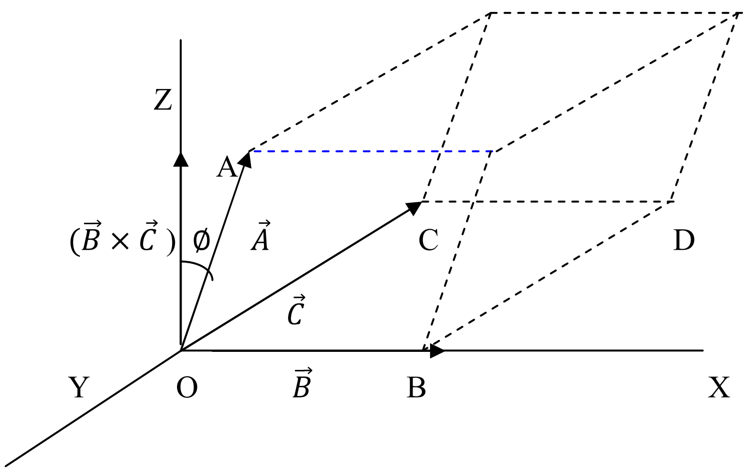


Figure 1.14

Therefore, it is clear that $\vec{A} \,.\, (\vec{B} \times \vec{C})$ represents the volume of parallelepiped constructed by vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ as its sides. Further, it is a scalar quantity as volume is scalar. It can also be noted that in case of scalar triple product the final product (volume of parallelepiped) remains same if the position of $\vec{A}, \vec{B}$ and $\vec{C}$ or dot and cross are interchanged.

$$[\vec{A}\,\vec{B}\,\vec{C}] = \vec{A}.(\vec{B}\times\vec{C}) = \vec{B}.(\vec{C}\times\vec{A}) = \vec{C}.(\vec{A}\times\vec{B}) = (\vec{B}\times\vec{C}).\vec{A} = (\vec{C}\times\vec{A}).\vec{B} = (\vec{A}\times\vec{B}).\vec{C}$$

Scalar triple product can also be explained by determinant as

$$[\vec{A}\,\vec{B}\,\vec{C}] = \vec{A}.(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

In case of three vectors to be coplanar, it is not possible to construct a parallelepiped by using such three vectors as its sides; therefore the scalar triple product must be zero.

$$[\vec{A}\,\vec{B}\,\vec{C}] = \vec{A}.(\vec{B}\times\vec{C}) = 0$$

1.3.3.2 Vector triple product:

The vector triple product of three vectors is define as

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$$

The vector triple product is product of a vector with the product of two another vectors. The vector triple product can be evaluated by determinant method as given below.

$$(\vec{B}\times\vec{C}) = \begin{vmatrix} i & j & k \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= i(B_yC_z - B_zC_y) - j(B_xC_z - B_zC_x) + k(B_xC_y - B_yC_x)$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix}$$

$$= (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$$

As in cross product the vector $\vec{A}\times(\vec{B}\times\vec{C})$ will be perpendicular to plane containing vectors $\vec{A}$ and $(\vec{B}\times\vec{C})$. Since $(\vec{B}\times\vec{C})$ is itself in the direction perpendicular to plane containing $\vec{B}$ and $\vec{C}$, therefore the direction of $\vec{A}\times(\vec{B}\times\vec{C})$ will be along the plan containing $\vec{B}$ and $\vec{C}$, $hence\ is\ a\ linear$ combination of $\overrightarrow{B}\ and\ \overrightarrow{C}.$

1.4 Summary

1. Physical quantities are of two types, scalar and vector. The scalar quantities have magnitude only but no direction. The vector quantities have magnitude as well as direction.
2. Two vector quantities can be added with parallelogram law and triangle law. In parallelogram law, the resultant is denoted by the diagonal of parallelogram whose adjacent sides are represented by two vectors. In triangle law, we place the tail of second vector on the head of first vector, and resultant is obtained by a vector whose head is at the head of second vector and tail is at the tail of first vector.
3. For subtraction, we reverse the direction of second vector and add it with first vector.
4. In case of more than two vectors we simply use Polygon law of vector addition.
5. Any vector can be resolved into two or more components. By adding all components we can find the final vector.
6. If a vector makes angles α, β and γ with three mutual perpendicular axes x, y and z respectively then $cos\ \alpha$, $cos\ \beta$ and $cos\ \gamma$ are called direction cosines.
7. Scalar product of two vectors is defined as $\overrightarrow{P}.\ \vec{Q}\ = PQ\cos\theta$ which is a scalar quantity.
8. Vector product of two vectors is defined as $\vec{A}\times\vec{B} = ABsin\emptyset\ \hat{n}$ which is a vector quantity. The direction of vector is perpendicular to $\vec{A}$ and $\vec{B}$.
9. If two vectors are parallel to each other then they are said to be collinear. For collinear vectors $\overrightarrow{P}.\ \vec{Q}\ = PQ$ or $\vec{P}\times\vec{Q} = 0$
10. If the angle between two vectors is 90^0, then vectors are called orthogonal. In this case $\overrightarrow{P}.\ \vec{Q}\ = 0$
11. Cross product of two vectors can also be calculated by determinant. The determinant form of cross product is

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

12. Scalar triple product of three vectors can also be calculated by determinant. The determinant form of Scalar triple product is

$$\overrightarrow{A}.(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

13. Vector triple product is defined as

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \left(\vec{A}.\vec{C}\right)\vec{B} - (\vec{A}.\vec{B})\vec{C}$$

1.5 Glossary

Vector- Physical quantity with direction
Scalar quantities- Physical quantity without direction
Collinear – in same line or direction
Orthogonal- perpendicular to each other
Coplanar – on same plane

1.6 Reference Books

1. Mechanics – D.S. Mathur, S Chand, Delhi
2. Concept of Physics- H C Verma, Bharti Bhawan, Patna
3. Physics Part-I, Robert Resnick and David Halliday, Wiley Eastern Ltd

1.7 Suggested readings

1. Modern Physics, Beiser, Tata McGraw Hill
2. Fundamental University Physics-I, M. Alonslo and E Finn, Addition-Wesley Publication
3. Berkeley Physics Course Vol I, Mechanics, C Kittel et al, McGraw- Hill Company

1.8 Terminal questions

1.8.1 Short answer type questions

1. Define unit vector, like vector and equal vectors.
2. What are direction cosines? Give its significance.
3. What angle does the vector $3i + \sqrt{2}j + k$ make with y axis?
4. What is the condition for vector to be collinear?
5. Explain the difference between dot and cross products.
6. What is angular momentum? How the direction of angular momentum can be decided?
7. Give some examples of dot product in physics.
8. Give some examples of cross product in physics.
9. Define scalar triple product.
10. How the angle between two vectors can be obtained?

1.8.2 Essay type questions

1. If $|\boldsymbol{A} + \boldsymbol{B}| = |\boldsymbol{A} - \boldsymbol{B}|$, show that **A** and **B** are perpendicular to each other.
2. What is the significance of dot product? Give the properties of cross product.
3. Show that $A = 5i + 2j + 4k \; and \; B = 2i + 3j - 4k$ are perpendicular to each other.
4. What is the vector product? Give the properties of vector product.
5. Find out the condition if two vectors are collinear.

6. Find the components of a vector along and perpendicular to the direction of another vector.

1.8.3 Numerical question

1. Calculate the dot product of vectors $\boldsymbol{A} = 6i + 7j + k \ and \ \boldsymbol{B} = i + 3j + 2k$.
2. A particle moves from the position $(3i + 3j + 2k) meter$ to another position $(-2i + 2j + 4k)\ meter$ under the influence of a force $\boldsymbol{F} = 3i + 2j + 4k$ newton. Calculate the work done by the force.
3. Obtain the projection of a vector $\boldsymbol{A} = 3i + 4j + 5k$ along a line which originates at a point (2, 2, 0) and passing through another point (-2, 4, 4).
4. Find the unit vector in the direction of resultant vectors of $\boldsymbol{A} = 6i + 7j + k \ and \ \boldsymbol{B} = i + 3j + 2k$.

UNIT 2: VECTOR CALCULUS

STRUCTURE:

2.0 Objective:

In previous unit we studied about the basic concepts of vector like its meaning, significance, representation, addition, subtraction etc. Now in this unit, we will learn some further use of vectors in physics and mathematics. After reading this unit we will able to understand:

1. Differentiation of vector
2. Del operator
3. Scalar and vector fields
4. Gradient
5. Curl
6. Divergence
7. Vector identities
8. Applications in physics

2.1 Introduction:

Differentiation and integration techniques are frequently used in physics and mathematics. Therefore this unit is basically vector calculus. Theses calculus techniques are used to solve and explain many physical problems. In this unit we will understand the differentiation and integration of vector quantities. Further we define some new terms like gradient, curl, divergence, its properties and application. The physical significance of these terms will also be discussed in detail.

2.2 Differentiation of vector:

Suppose $\vec{r}$ is the position vector of a particle situated at point P with respect to origin O. If particle moves with time, then vector $\vec{r}$ varies corresponding to time t, and $\vec{r}$ is said to be vector function of scalar variable t and represented as $\vec{r}$ =F(t)

If P is the position of particle at time t then OP = $\vec{r}$

If Q is the position of particle at time t+δt and position vector of Q is $(\vec{r} + \delta\vec{r})$

then $\overrightarrow{PQ} = \overrightarrow{OQ}$-$\overrightarrow{OP}$

$$= \vec{r} + \delta\vec{r} - \vec{r}$$

In limiting case if $\delta t \to 0$ then $\delta\vec{r} \to 0$ and P tends to Q and the chord become the tangent at P. Differentiation is define as

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{r}(t + \delta t) - \vec{r}(t)}{\delta t}$$

When the limit exists only then the function $\vec{r}$ is differentiable. If we further differentiate function with respect t then it is called second order differentiation. If should be cleared that the derivatives of a vector (say $\vec{r}$) are also vector quantities.

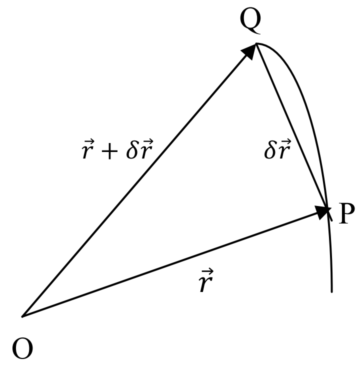


Figure 2.1

2.2.1 Properties of vector differentiation:

If $\vec{A}$ and $\vec{B}$ are two vectors, $\emptyset$ is a scalar field and $\vec{C}$ is a constant vector then

(1) $\frac{d}{dt}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$

(2) $\frac{d}{dt}(A \times \emptyset) = \frac{d\vec{A}}{dt}\emptyset + \vec{A}\frac{d\emptyset}{dt}$

(3) $\frac{d}{dt}(\vec{A}.\vec{B}) = \vec{A}.\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}.\vec{B}$

(4) $\frac{d}{dt}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$

(5) $\frac{d\vec{C}}{dt} = 0$

(6) $\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$ if s is scalar function of t.

(7) $\frac{d}{dt}(r^2) = \frac{d}{dt}(\vec{r}.\vec{r}) = \vec{r}\frac{d\vec{r}}{dt} + \vec{r}\frac{d\vec{r}}{dt} = 2\vec{r}\frac{d\vec{r}}{dt}$, if $\vec{r}$ is position vector.

Example 2.1: A particle is moving along the curve x $= t^2 + 2$, y$= t^2 + 1$ and z$= 3t + 5$. Find the velocity and acceleration of particle along the direction 3i+2j+6k at time t=2.

Solution:
Curve is define as x $= t^2 + 2$, y$= t^2 + 1$ and z$= 3t + 5$.
The position vector of particle at any time t is given as

$$\bar{r} = xi + yj + zk$$

$$\bar{r} = (t^2 + 2)i + (t^2 + 1)j + (3t + 5)k$$

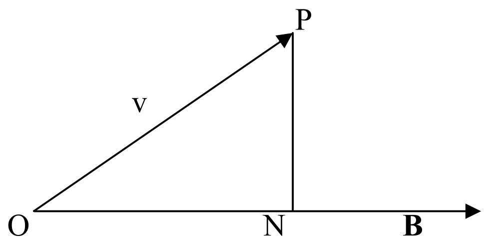


Figure 2.2

Velocity is given as

$$\frac{d\bar{r}}{dt} = 3t^2 i + 2tj + 3k$$

at t=2 velocity becomes

$$\frac{d\bar{r}}{dt} = 12i + 4j + 3k$$

Component of the velocity along the direction $3i + 2j + 6k = \vec{B}\ (say\ )$

$$ON = |\bar{v}| \cos\theta \,.\, \hat{b} = |\bar{v}|\ \frac{\bar{v}.\bar{B}}{|\bar{v}||\bar{B}|} . \frac{B}{|\bar{B}|} = \frac{(\bar{v}.\bar{B})B}{|B|^2}$$

$$= \frac{(16\ i+4j+3k).(3i+2j+6k)}{3^2+2^2+6^2}(3i + 2j + 6k) = \frac{74}{49}(3i + 2j + 6k)$$

acceleration $\bar{a}$ can be given as $\bar{a} = \frac{d\bar{r}}{dt} = 6t\ i + 2j$

acceleration $\bar{a}$ at t=2 can be given as $\bar{a} = 12i + 2j$

Component of acceleration along direction $\bar{B}$ is given as

$$= |\bar{a}| \cos\theta \,.\, \hat{b} = |\bar{a}| \frac{\bar{a}.\bar{B}}{|\bar{a}|\ |B|} \frac{\bar{B}}{|B|} = \frac{(\bar{a}.\bar{B})\ \bar{B}}{|B|^2}$$

$$= \frac{(12i+2j).(3i+2j+6k)}{32^2+2^2+6^2}(3i + 2j + 6k)$$

$$= \frac{52}{49}(3i + 2j + 6k)$$

2.2.2 Partial derivative:

If f is a vector function which depends on variable (x, y, z), then the partial derivatives are defined as

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y, z) - f(x, y, z)}{\delta x}$$

$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y, z) - f(x, y, z)}{\delta y}$$

$$\frac{\partial f}{\partial z} = \lim_{\delta z \to 0} \frac{f(x, y, z + \delta z) - f(x, y, z)}{\delta z}$$

In case of partial derivatives with respect to a variable, all the other remaining variables are taken as constant.

Partial derivatives of second order are defined as:

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} (\frac{\partial f}{\partial x})$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} (\frac{\partial f}{\partial y})$

$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z} (\frac{\partial f}{\partial z})$

2.2.3 Del operator:

The vector differential operator del is denoted by ∇ and is defined as

$$\nabla = \mathrm{i} \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$$

2.2.4 Scalar and vector point functions:

(1) Field: Field is a region of the space defined by a function.

(ii) Scalar point function: A scalar function $\emptyset(x, y, z)$ defines all scalar point in the space. For example, gravitational potential is a scalar function defined at all gravitational fields in the space.

(iii) Vector potential function: If a vector function $\vec{F}$(x, y, z) defines a vector at every point in space then it is called vector point function. For example gravitational force is a vector function defined at a gravitational field in the space.

2.2.5 Gradient:

The gradient of a scalar function $\emptyset$ is defined as

grad $\emptyset = \nabla\emptyset = (\mathrm{i} \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z})\emptyset$

$= \mathrm{i} \frac{\partial \emptyset}{\partial x} + j \frac{\partial \emptyset}{\partial y} + k \frac{\partial \emptyset}{\partial z}$

grad $\emptyset$ $is\ a\ vector\ qunatity.$

Total differential d$\emptyset$ of a scalar function $\emptyset(x, y, z)$ can be expressed as,

$$d\emptyset = \frac{\partial \emptyset}{\partial x}dx + \frac{\partial \emptyset}{\partial y}dy + \frac{\partial \emptyset}{\partial z}dz$$

Total differential d∅ of a scalar function ∅ can also be expressed as

d∅ $= \frac{\partial \emptyset}{\partial x}dx + \frac{\partial \emptyset}{\partial y}dy + \frac{\partial \emptyset}{\partial z}dz$

$= (i\frac{\partial \emptyset}{\partial x} + j\frac{\partial \emptyset}{\partial y} + k\frac{\partial \emptyset}{\partial z})\,(idx + jdy + kdz)$

$d\emptyset = (\vec{\nabla}\emptyset).\vec{dr} = |\nabla\emptyset||dr|\cos\theta = (\vec{\nabla}\emptyset).\ dr\,\hat{r}$, (where $\hat{r}$ is a unit vector along $d\vec{r}$)

also θ is angle between $\vec{\nabla}\emptyset$ and $d\vec{r}$ (The direction of displacement).

So, $\frac{d\emptyset}{dr} = (\vec{\nabla}\emptyset).\ \hat{r}$

Thus , $\frac{d\emptyset}{dr}$ is the directional derevative of ∅. The rate of change is maximum if $\hat{r}$ is along $\vec{\nabla}\emptyset$ i.e. angle between $\vec{\nabla}\emptyset$ and $\hat{r}$ is zero.

Hence gradient of the scalar field ∅ defines a vector field, the magnitude of which is equal to the maximum rate of change of ∅ and the direction of which is the same, as the direction of displacement along with the rate of change is maximum.

Example 2.2:
In the heat transfer, the temperature of any point in space is given by $T = xy + yz + zx$. Find the gradient of T in the direction of vector 4i-3k at a point (2, 2, 2).

Solution:

Temperature is define as

$T = xy + yz + zx$

gradient of temperature T is given as

$$grad\ T = \nabla T = \left(i\frac{\partial}{\partial x} + j\ \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\partial\right)(xy + yz + zx)$$

$$\nabla T = i(y + z) + j(x + z) + k(x + y)$$

at point (2, 2, 2) the ∇T is $(4i + 4j + 4k)$

The gradient T in the direction of vector 4i-3k is

$= (4i + 4j + 4k)$. Unit vector along $(4i - 3k)$

$=(4i+4j+4k).\frac{(4i-3k)}{\sqrt{4^2+3^2}}$

=4/5

2.26 Physical significance of grad $\emptyset$:

The physical significance of grad $\emptyset$ can be explained on the basis of surface defined by scalar field $\emptyset$. The value of $\emptyset$ remains constant on the surface S, as shown in figure 2.3 and it is called a level surface or equi-scalar surface. Let us consider two surfaces S and S' defined by scalar function $\emptyset$ and $\emptyset$+d$\emptyset$ respectively. Suppose $\vec{n}$ is normal to the surfaces S and S'. If the coordinates of point P and Q are (x, y, z) and (x+dx, y+dy, z+dz) then the distance between P and Q are

$$d\vec{r} = idx + jdy + kdz$$

as the definition of differentiation

$$d\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz$$

$$= \left(\frac{\partial\emptyset}{\partial x}i + \frac{\partial\emptyset}{\partial y}y + \frac{\partial\emptyset}{\partial z}k\right).(dx\, i + dy\, j + dz\, k)$$

$$d\emptyset = \vec{\nabla}\emptyset.d\vec{r}$$

If we consider the point Q approaches to P and finally lies on P then

$$d\emptyset = 0$$

$$\vec{\nabla}\emptyset.d\vec{r} = 0$$

$\nabla\emptyset$ *and dr are perpendiular to each other.*

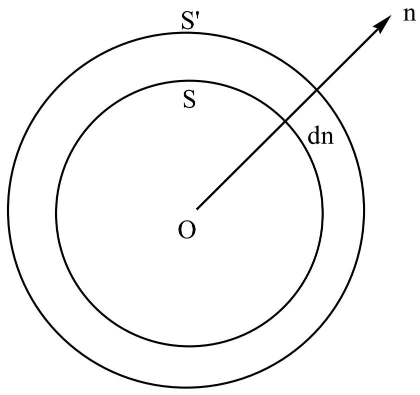


Figure 2.3

Therefore, $\nabla\emptyset$ is a vector which is perpendicular to the surface S.

If $\vec{n}$ is normal on the surface S and d$\vec{n}$ represents the distance between surfaces S to S' then $dn = dr\ cos\ \theta = \hat{n}.d\vec{r}$

And $d\emptyset = \frac{\partial\emptyset}{\partial n}dn = \frac{\partial\emptyset}{\partial n}\hat{n}.d\vec{r}$

By using equation (1), $\vec{\nabla}\emptyset.d\vec{r} = \frac{\partial\emptyset}{\partial n}\hat{n}.d\vec{r}$

$$\vec{\nabla}\emptyset = \frac{\partial\emptyset}{\partial n}\hat{n}$$

Thus, $\nabla\emptyset$ is defined as a vector whose magnitude is rate of change of $\emptyset$ along normal to the surface and direction is along the normal to the surface.

Example2.3:

Find the directional derivative of a scalar function $\emptyset(x,y,z) = x^2 +$ xy + z ^{2} at the point A (2,-1,-1) in the direction of the line AB where coordinate of B are (3, 2, 1).

Solution:

The component of $\nabla\emptyset$ along the direction of a vector $\vec{A}$ is called directional derivative of $\emptyset$ and given as $\nabla\emptyset\ .\hat{A}$

Now $\nabla\emptyset = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + xy + z^2)$

$=(2x + y)i + x\ j + 2z\ k$

gradient at point A (2,-1,-1)

$\nabla\emptyset = 3i + 2\ j - 2\ k$

The vector $\overrightarrow{AB} = position\ vector\ of\ B - position\ vector\ of\ A$

$= (3i + 2j + k) - (2i - j - k) = i + 3j$

Directional derivative of $\emptyset$ in the direction of AB is

$$\vec{\nabla}\emptyset\,.\widehat{AB} = (3i + 2j - 2k).\frac{(i+3j)}{\sqrt{1+9}} = \frac{9}{\sqrt{10}}$$

2.3 Divergence of Vector:

The divergence is defined as dot product of del operator with any vector point function $\vec{f}$ or any vector $\bar{F}$ and given as,

div. $\vec{f} = \nabla.\vec{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\,(if_x + jf_{y+}kf_z)$ where $\vec{f} = \ if_x + jf_{y+}kf_z$

$$= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

Since divergence of a vector $\vec{f}$ is dot product of del operator $\vec{\nabla}$ and that vector $\vec{f}$, therefore it is a scalar quantity.

2.3.1 Physical Significance of Divergence:

On the basis of fluid dynamics or a fluid flow, the divergence of a vector quantity can be explained. Let us consider a parallelepiped of edges dx, dy and dz along the x, y, z directions as shown in figure 2.4.

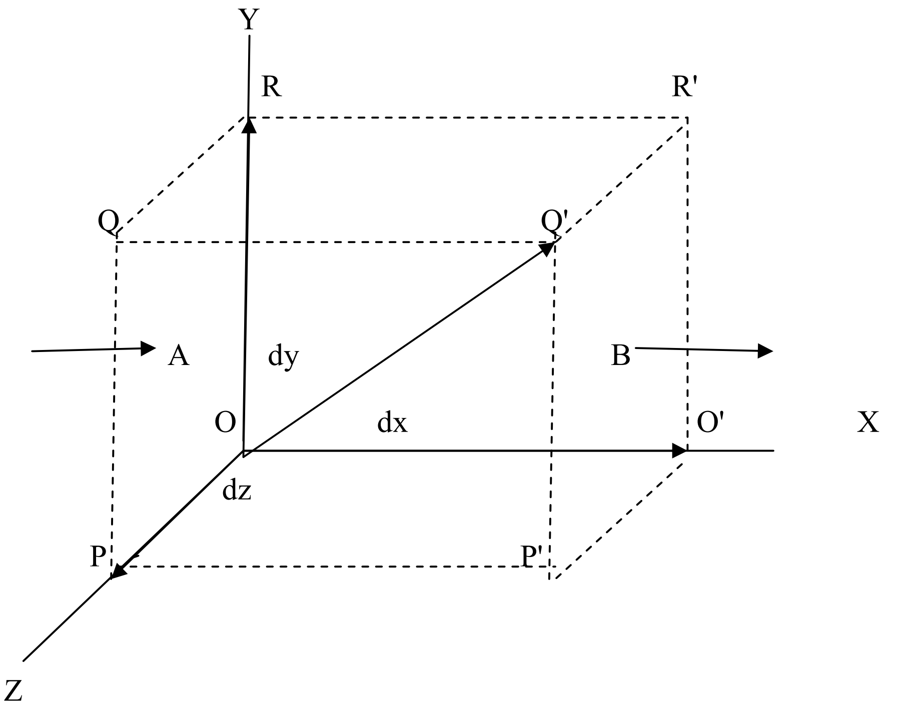


Figure 2.4

Let $\vec{v}$ is the velocity of fluid at A(x, y, z) and given as

$$\vec{v} = v_x i + v_y j + v_z k$$

Where the $v_{x,}v_y, v_z$ *are the components of veolcity along* x, y, z *directions.*

Amount of fluid entering through the surface O'P'Q'R' per unit time is given as:

$$velocity \times area = v_x dydz$$

Amount of fluid flowing out through the surface O'P'Q'R' per unit times is given as

$= v_{x+dx}\, dydz$

$=(v_x + \frac{\partial v_x}{\partial x} dx)dydz$

Decrease in the amount of fluid in the parallelepiped along x axis per unit time.

$=v_x dydz - \left(v_x + \frac{\partial v_x}{\partial x} dx\right) dydz$

$= -\frac{\partial v_x}{\partial x} dxdydz$

Negative sign shows, decrease in the amount of fluid inside the parallelepiped.

Similarly decrease of amount of fluid along y axis

$=-\frac{\partial v_y}{\partial y} dxdydz$

Decrease of amount of fluid along z axis

$= -\frac{\partial v_z}{\partial z} dxdydz$

Total amount of fluid decrease inside the parallelepiped per unit time$= -\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dxdydz)$

Thus the rate of loss of fluid per unit volume $= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

(We can ignore negative sign when we specify that the negative sign indicates decrease in the amount of fluid).

Further the rate of loss of fluid per unit volume

$=\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(v_x i + v_y j + v_z k) = \vec{\nabla}.\vec{v} = div\ \vec{v}$

Thus, *the divergence of velocity vector shows the rate of loss of fluid per unit timer per unit volume.*

If we consider fluid is incompressible, there is not any loss or gain in the amount of fluid, therefore $div\ v = 0$

If the divergence of a vector is 0, then the vector function is called solenoidal.

Example 2.4: if u=$x^2+y^2+z^2$ and $\bar{r} =$ 2xi + 3yj + 2zk, then find the div (u$\bar{r}$).

Solution : Div (u $\bar{r}$) = $\nabla.$(u $\bar{r}$)

$$\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).[(x^2+y^2+z^2)(2xi+3yj+2zk)]$$

$$= i\frac{\partial}{\partial x}(x^2 2x)i + j\frac{\partial}{\partial y}(y^2 3y)j + k\frac{\partial}{\partial z}(z^2.2z)k$$

$$= 6x^2+9y^2+6z^2$$

2.4 Curl

The curl of a vector $\vec{F}$ is defined as

Curl $\overline{F} = \nabla\times\overline{F}$ (where $\overline{F}$=$F_x i + F_y j + F_j k$)

$=(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z})\times(F_x i+F_y j+F_j k)$

In terms of determinant of vector product

Curl $\overline{F}$=$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

Since curl is vector product of two vectors, therefore it is a vector quantity.

2.4.1 Physical significance of curl:

On the basis of angular velocity and linear velocity the curl can be explained.

Let us consider a particle moving with velocity $\overline{v}$ and $\overline{r}$ is the position vector of particle rotating around origin O. Let $\vec{\omega}$ is the angular velocity of particle then

$$curl\,\overline{v} = \nabla\times\overline{v}$$

$$= \nabla\times(\overline{\omega}\times\overline{r}) \qquad (\because\ \overline{v}=\overline{\omega}\times\overline{r})$$

$$= \nabla(\omega_x i+\omega_y j+\omega_z k)\times(xi+yj+zk)$$

$$= \nabla\times\begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[\left(\omega_y z - \omega_z y\right)i - (\omega_x z - \omega_z x)j + (\omega_x y - \omega_y x)k\right]$$

$$= \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_y z - \omega_z y & \omega_z x - \omega_x z & \omega_x y - \omega_y x \end{bmatrix}$$

$$curl\, \overline{v} = 2\left(\omega_x i + \omega_y j + \omega_z k\right) = 2\overline{\omega}$$

Thus the curl of velocity shows angular velocity which means rotation of particle. Thus curl of a vector quantity is connected with rotational properties of vector field. If curl of a vector is zero, $\nabla \times \overline{f} = 0$ then there is no rotational property and $\overline{f}$ is called irrotational.

Example 2.5

Calculate the curl of a vector given by $\overline{F} = xyzi + 2x^2yj + (x^2z^2 - 2y^2)$k.

Solution:

$curl\, \overline{F} = \nabla \times \overline{F}$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (xyzi + 2x^2yj + (x^2z^2 - 2y^2)k)$$

$$= \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 2x^2y & x^2z^2 - 2y^2 \end{bmatrix}$$

$$= -4yi - (2xz^2 - xy)j + (4xy - xz)k$$

Example2.6:

Show that $F = (y^2 + 2xz^2)i + (2xy - z)j + (2x^2z - y + 2z)\vec{k}\ is\ irrolational.$

Solution:

$curl\, F = \nabla \times F$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times [(y^2 + 2xz^2)i + (2xy - z)j + (2x^2z - y + 2z)\vec{k}]$$

$$= 0$$

Therefore F is irrotational.

2.5 Vector integral:

2.5.1 Line Integral: The integral of a vector function $\vec{F}$ along a line or curve is called line integral.

Suppose $\vec{F}$(x, y, z) be a vector function and PQ is a curve and $\overrightarrow{dl}$ is a small length of curve then line integral of vector $\vec{F}$ along a length $\overrightarrow{dl}$ is given as

$\int_l \vec{F}.d\vec{l}$

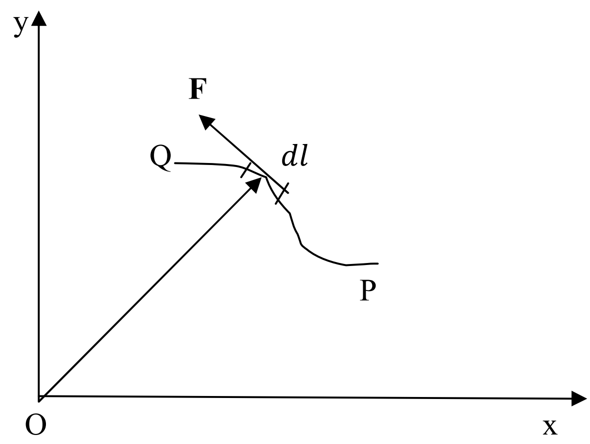


Figure 2.5

The integral may be closed or open depending on the nature of the curve whether closed or open. To compute the line integral of a function F, any method of integral calculus may be employed. In case of fore $\vec{F}$ acting on a particle along a curve PQ, the total work done can be calculated as line integral of force.

Work done= $\int_p^Q \vec{F}.\overrightarrow{dl}$

2.5.2 Surface integral:

Similarly as line integral of F is a vector function and s is a surface, then surface integral of a vector function F over the surface s is given as

Surface integral=$\iint_s \vec{F}.\overrightarrow{dl}$

The direction of surface integral is taken as perpendicular to the surface s.

If ds is written as ds=dxdy

Surface integral=$\iint_s \vec{F}.d\vec{s} = \int_x \int_y F.dxdy$

Surface integral represents flux through the surface S.

2.5.3 Volume integral:
If dV denotes the volume defined by $dxdydz$ then the volume integration of a vector F is define as

Volume integral=$\int_V FdV = \int_x \int_y \int_z F.dxdydz$

The volume integral can be explained in terms of total charge inside a volume. Suppose ρ is charge density of a volume dV then total charge inside the volume is given as q=$\int_v \rho\, dV$

2.6 Vector identities:
If $\emptyset_1$ and $\emptyset_2$ are two scalar point functions and $\vec{A}$ and $\vec{B}$ are two vectors, then

$$\nabla(\emptyset_1 + \emptyset_2) = \nabla\emptyset_1 + \nabla\emptyset_2$$

$$\nabla(\emptyset_1\emptyset_2) = \emptyset_1\nabla\emptyset_2 + \emptyset_2\nabla\emptyset_1$$

$$div\left(\vec{A} + \vec{B}\right) = div\,\vec{A} + div\,\vec{B}$$

$$div\left(\vec{A}.\vec{B}\right) = \overrightarrow{A.}\ div\,B + \overrightarrow{B.}\ div\,\vec{A}$$

$$curl\left(\vec{A} + \vec{B}\right) = curl\,\vec{A} + curl\,\vec{B}$$

$$div\left(\emptyset\,\vec{A}\right) = \emptyset\,div\,\vec{A} + \vec{A}\,.grad\,\emptyset$$

$$curl\left(\emptyset\vec{A}\right) = \emptyset\,curl\,\vec{A} + grad\,\emptyset\ \times \vec{A}$$

$$div\,curl\,\vec{A} = 0$$

$$curl\,grad\ \ \emptyset = 0$$

$$div\left(\vec{A} \times \vec{B}\right) = \vec{B}.curl\,\vec{A} + \vec{A}.curl\,\vec{B}$$

$$curl\,curl\,\vec{A} = grad\,div\,\vec{A} - \nabla^2\,\vec{A}$$

Example2.7 : Prove that

(1) $div\,curl\,\vec{A} = 0$
(2) $curl\,grad\ \ \emptyset = 0$

Solution:

(1) (1) $div\,curl\,\vec{A} = \nabla.\nabla \times \vec{A}$

$$= \nabla.\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$=\nabla.\left[i\ \left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+j(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})+\mathrm{k}(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})\right]$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\frac{\partial}{\partial y}(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})+\frac{\partial}{\partial z}(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})$$

$$=0$$

(2) $curl\ grad\ \emptyset = \nabla \times \nabla\emptyset$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\emptyset}{\partial x} & \frac{\partial\emptyset}{\partial y} & \frac{\partial\emptyset}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2\emptyset}{\partial y\partial z}-\frac{\partial^2\emptyset}{\partial z\partial y}\right)+j\left(\frac{\partial^2\ \emptyset}{\partial z\partial x}-\frac{\partial^2\emptyset}{\partial x\partial z}\right)+k\left(\frac{\partial^2\emptyset}{\partial x\partial y}-\frac{\partial^2\emptyset}{\partial y\partial x}\right)=0$$

Example2.8:

Show that

(i) $div\left(\vec{A}\times\vec{B}\right)=\vec{B}.curl\ \vec{A}-\vec{A}\ .curl\ \vec{B}$

(ii) $curl\ curl\ \vec{A}=grad\ div\ \vec{A}-\nabla^2\ \vec{A}$

Solution (i) $div\left(\vec{A}\times\vec{B}\right)=\ \nabla.(\vec{A}\times\vec{B})$

$$= (\hat{\imath}\ \frac{\partial}{\partial x}+\hat{\jmath}\ \frac{\partial}{\partial y}+\hat{k}\ \frac{\partial}{\partial z}).\ [(A_yB_z-A_zB_y)\ \hat{\imath}+(A_zB_x-A_xB_z)\ \hat{\jmath}+(A_x\ B_y-A_yB_x)\hat{k}]$$

$$= \frac{\partial}{\partial x}(A_yB_z-A_zB_y)+\frac{\partial}{\partial y}(A_zB_x-A_xB_z)+\frac{\partial}{\partial z}(A_xB_y-A_yB_x)$$

$$= B_x(\frac{\partial Az}{\partial y}-\frac{\partial Ay}{\partial z})+B_y(\frac{\partial Ax}{\partial z}-\frac{\partial Az}{\partial x})+B_z(\frac{\partial Ay}{\partial x}-\frac{\partial Ax}{\partial y})-A_x(\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z})-A_y(\frac{\partial Bx}{\partial z}-\frac{\partial Bz}{\partial x})$$

$$-A_z(\frac{\partial By}{\partial x}-\frac{\partial Bx}{\partial y})$$

$$= (B_x\hat{\imath}+B_y\hat{\jmath}+Bz\hat{k}).\ [(\frac{\partial Az}{\partial y}-\frac{\partial Ay}{\partial z})\hat{\imath}+(\frac{\partial Ax}{\partial z}-\frac{\partial Az}{\partial x})\hat{\jmath}+(\frac{\partial Ay}{\partial x}-\frac{\partial Ax}{\partial y})\hat{k}]-$$

$$(A_x\hat{\imath}+A_y\hat{\jmath}+Az\hat{k}).\ [(\frac{\partial Bz}{\partial y}-\frac{\partial By}{\partial z})\hat{\imath}+(\frac{\partial Bx}{\partial z}-\frac{\partial Bz}{\partial x})\hat{\jmath}+(\frac{\partial By}{\partial x}-\frac{\partial Bx}{\partial y})\hat{k}]$$

$$= \vec{B}.\ \mathrm{curl}\ \vec{A}-\vec{A}.\ \mathrm{curl}\ \vec{B}$$

= curl $\vec{A}$. $\vec{B}$ - curl $\vec{B}$. $\vec{A}$

Solution (ii)

$$curl\ curl\ \bar{A} = \nabla \times (\nabla \times \bar{A})$$

$$= (i\frac{\partial}{\partial x} + j\ \frac{\partial}{\partial y} + k\frac{\partial}{\partial z}) \times \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= (i\frac{\partial}{\partial x} + j\ \frac{\partial}{\partial y} + k\frac{\partial}{\partial z}) \times \left[i\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - j(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \mathrm{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_y}{\partial y} - \frac{\partial A_y}{\partial z}\right) & (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) & (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\ \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\right] + j\left[\frac{\partial}{\partial z}\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial x}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\right]$$
$$+k\left[\frac{\partial}{\partial x}\ \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\right]$$

$$= i\left[\frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial x}\right] + j\left[\frac{\partial^2 A_z}{\partial z \partial y} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_x}{\partial x \partial y}\right]$$
$$+k\left[\frac{\partial^2 A_x}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial y \partial z}\right]$$

$$= \sum i\left[\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y \partial x} + \frac{\partial^2 A_z}{\partial z \partial x}\right) - \ (\frac{\partial^2 A_x}{\partial x^2} + \ \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2})\right]$$

$$= \sum i\frac{\partial}{\partial x}\ \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \sum i\left[\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\ \right)\right]$$

$$= grad\ div\ \bar{A} - \nabla^2\ \bar{A}$$

2.7 Summary:

1. Differentiation and integration techniques are used to solve and explain many physical problems. Differentiation of a vector is defined as

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t}$$

2. If we further differentiate function with respect t then it is called second order differentiation. If should be cleared that the derivatives of a vector (say $\vec{r}$) are also vector quantities. If r is a position vector of a particle at time t then $\frac{d\vec{r}}{dt}$ denotes its velocity.

3. Partial derivative is defined as

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y, z) - f(x, y, z)}{\delta x}$$

In case of partial derivative with respect to a variable, all the other remaining variables are taken as constant.

4. Vector differential operator del is denoted by ∇ and defined as

$$\nabla = \mathrm{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

5. The gradient of a scalar function ∅ is defined as

grad $\emptyset = \nabla\emptyset = (\mathrm{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z})\emptyset$

6. The divergence is dot product of del operator with any vector point function $\overrightarrow{f}$ and is given as

div. $\vec{f} = \nabla.\vec{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(if_x + jf_{y+}kf_z)$ where $\vec{f} = if_x + jf_{y+}kf_z$

$$= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

7. The curl of a vector $\vec{F} = F_x i + F_y j + F_j k$ is defined as

Curl $\overline{F} = \nabla \times \vec{F} = (i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}) \times (F_x i + F_y j + F_j k)$

8. The integral of a vector function $\vec{F}$ along a line or curve is called line integral and given as $\int_l \vec{F}.\overrightarrow{dl}$

9. If $\vec{F}$ is a vector function and s is a surface, then surface integral of a vector function $\vec{F}$ over the surface S is given as $\iint_s \vec{F}.d\vec{s}$

10. If dV denotes the volume defined by dxdydz then the volume integration of a vector F is defined as $\int_V FdV = \int_x \int_y \int_z F.dxdydz$

2.8 Glossary:

Displacement – net change in location of a moving body.

Differentiation- instantaneous rate of change of a function with respect to one of its variables Integration- The process of finding a function from its derivative. (Reverse of differentiation)

Partial derivative- derivative of a function with respect to a variable, if all other remaining variables are considered as constant

Operator – An *Operator* is a symbol that shows a mathematical operation.

del operator - vector differentiation operator

gradient- derivative of function.(rate of change of a function or slope)

divergence- rate at which density exits at a given region of space. (flux density)

Curl- describes the rotation of vector field.

line integral- Integration along a line.

surface integral- Integration along a surface.

volume integral- Integration along a volume.

1.9 **Self Assessment Question (SAQ):**

1. If $\emptyset(x, y, z) = (x^2 + y^2 + z^2)^{-5/2}$ then calculate $\nabla\emptyset$ at a point (1 1 1).
2. Calculate the gradient of a scalar function $\emptyset\ (x\ y\ z) = x^2 + y^2 + e^z$ at point (1, 2, -2).
3. If vector $\vec{B} = 3xyi + 5zj + 2yz^2k$ represents the magnetic field then calculate the flux at point (2, 2, 1).
4. Fiend the curl of a vector $\vec{A} = 3i + 5yzj + 5yz^2k.$
5. Given a vector function $\vec{F} = yi + xj$, calculate the line integration $\int_l \vec{F}.\overrightarrow{dl}$ from point (1,1,1) to (8, 2, -2) along the line joining these two points.
6. Show that $\nabla = 3y^4z^2i + 4x^3z^2j - 3x^2y^2k$ is a solenoidal vector.
7. Prove that $div\ grad\ \ \emptyset = \nabla^2\emptyset$
8. Prove that $div\ (\emptyset\ A) = \emptyset\ div\ A + A\ .grad\ \emptyset$
9. Explain the physical meaning of curl.
10. Explain different type of vector fields.

2.10 Reference Books:

1. Mechanics – D.S. Mathur, S Chand, Delhi
2. Concept of Physics- H C Verma, Bharti Bhawan, Patna
3. Physics Part-II, Robert Resnick and David Halliday, Wiley Eastern Ltd

2.11 Suggested readings:

1. Modern Physics, Beiser, Tata McGraw Hill
2. Fundamental University Physics-I, M. Alonslo and E Finn, Addition-Wesley Publication
3. Berkeley Physics Course Vol I, Mechanics, C Kittel et al, McGraw- Hill Company

2.12 Terminal questions:

2.12.1 Short answer type questions

1. Define gradient of a scalar function $\emptyset$.

2. Show that $\nabla\emptyset$ is a vector whose magnitude is equal to maximum rate of change of $\emptyset$ with respect to space variable.

3. Show that $\nabla\emptyset$ is perpendicular to surface $\emptyset$.

4. Solve $\nabla\left(\frac{1}{r}\right)$ for $r \neq 0$

5. If vector $\vec{F} = 6xzi - y^2\mathrm{j} + \mathrm{yzk}$ then calculate $\int_S \vec{F}.\hat{n}\, dS$ where S is the surface of a cube with boundaries $x = 0\ to\ x = 2$, y= $0\ to\ y = 2$, $z = 0\ to\ z = 2$.

6. Obtain the value $[grad\ \emptyset(\vec{r})] \times \vec{r}$

7. Find the area of parallelogram determined by the vectors $(i + 2\mathrm{j} + 3\mathrm{k})$ and $(-3i - 2\mathrm{j} + 4\mathrm{k})$.

Essay type questions

1. Define divergence of a vector function and its physical significance. Obtain the expression for the divergence of a vector $\vec{F}$.
2. Define curl of a vector function and its physical significance. Obtain the expression for the curl of a vector $\vec{F}$.
3. Prove that $\nabla \times (\vec{A} \times \vec{B}) = (\vec{\mathrm{B}}.\vec{\nabla})\vec{\mathrm{A}} - \overrightarrow{(\mathrm{A}.\nabla)}\vec{B} + \vec{A}\ div\ \vec{B} - \vec{B}\ div\vec{A}$
4. Prove that any vector function can be expressed as the sum of lamellar vector and solenoidal vector.
5. Derive the equation of continuity

$$\frac{\partial\rho}{\partial t} + \mathrm{div\,J} = 0$$

And show that how this equation express charge conservation.
6. Show that $\vec{u} \times \vec{v}$ is solenoidal if $\vec{u}$ and $\vec{v}$ are irrotational.

UNIT 3: GAUSS, STOKE and GREEN'S THEOREM

STRUCTURE:

3.0 Objective:

In this unit we will be able to understand the relation between surface integral and volume integral, line integral and surface integral etc. further, we can understand Laplace's equation and Poisson's equation.

3.1 Introduction:

In physics and mathematics many times, we transform one type of integral to another. These transforms are required for simplifying the problems. For example, if we are interested to calculate flux through a surface enclosed by a charge, and it is difficult to calculate flux by using surface integral, then it can be calculated by volume integral of charge inside the surface. Gauss's divergence theorem enables us to transform surface integral into volume integral. Similarly, Stoke's theorem transforms surface integral into line integral. In Green's Theorem we can transform two scalar functions simultaneously from volume to surface integral. By using these theorems, we can find easy approach to solve a problem.

3.2 Gauss divergence theorem:

Gauss divergence theorem is a relation between surface integration and volume integration. The theorem states:

The surface integral of a vector filed $\vec{F}$ over a closed surface s is equal to the volume integral of divergence of $\vec{F}$ taken over the volume enclosed by surface s.

Mathematically $\iint_s \vec{F}.d\vec{s} = \iiint_v div\,\vec{F}\,dv$

Mathematical proof:

Let us consider a vector $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$

According to Gauss divergence theorem $\iint_s (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}).ds = \iiint_v \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k})dxdydz$

Or $\iint_s (F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}).ds = \iiint_v \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz$ (1)

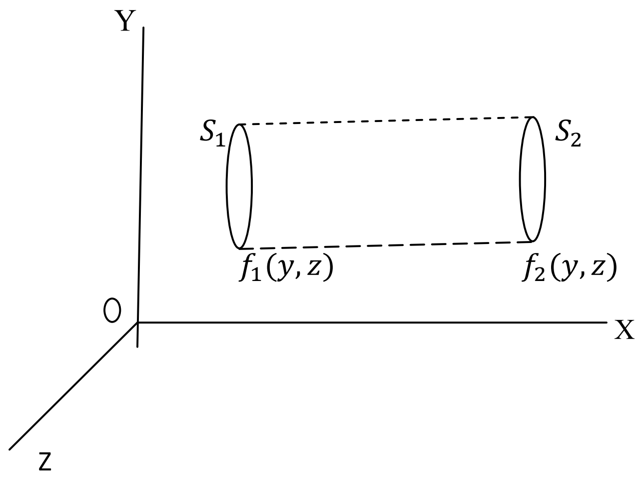


Figure 3.1

Now we can prove equation (1)

Let us first evaluate $\iiint_v \frac{\partial F_1}{\partial x} dxdydz = \iint_s \left[\int_{x=f_1(y,z)}^{x=f_2(y,z)} \frac{\partial F_1}{\partial x} dx\right] dydz$

$$= \iint_s [F_1(x,y,z)]_{x=f_1(y,z)}^{x=f_2(y,z)} dydz$$

$$= \iint_S [F_1(f_2,y,z) - F_1(f_1,y,z)]dydz \qquad (2)$$

Now, the right portion of surface i.e. S_2 can be given as

$dydz = \hat{n}_2.ids_2$ where $\hat{n}_2$ is the direction of unit vector perpendicular to the surface

Similarly the left portion of surface S_1 can be given as

$dydz = \hat{n}_1.ids_1$

Putting the value of area in the factors of RHS of equation (2) we have

$$\iint_s \left[F_1(f_2,y,z)dydz = +\iint_{S_2} F_1\hat{n}_2.i\ ds_2\right]$$

$$\iint_s \left[F_1(f_1,y,z)dydz = -\iint_{S_1} F_1\hat{n}_1.i\ ds_1\right]$$

Since the outward flux at surface S_2 is in the direction along the x axis and flux at surface S_1 is along the negative direction of x axis. Therefore, S_1 component is negative.

Putting the above values in equation (2) we have

$$\iiint_v \frac{\partial F_1}{\partial x} dxdydz = \iint_{S_2} F_1\hat{n}_{\ 2}.i\ ds_2 + \iint_{S_1} F_1\hat{n}_{\ 1}.i\ ds_2$$

$$\iiint_v \frac{\partial F_1}{\partial x} dv = \iint_S F_1\hat{n}.i\ ds$$

Since $\hat{n}_{\ 1}$ and $\hat{n}_{\ 2}$ are the direction perpendicular to yz plane that is along x axis shown by $\hat{n}$.

Similarly it can be shown that

$$\iiint_v \frac{\partial F_2}{\partial y} dv = \iint_s F_2 \hat{n}.j\, ds$$

And

$$\iiint_v \frac{\partial F_3}{\partial z} dv = \iint_s F_3 \hat{n}.k\, ds$$

Adding all above terms

$$\iiint_v \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\ dv = \iint_S (F_1\hat{n}.i + F_2\hat{n}.j + F_3\hat{n}.k)\ ds$$

Or $\iiint_v (\nabla.F)dv = \iint_s \vec{F}.d\vec{s}$

This is Gauss divergence theorem. The theorem relates the flux of a vector filed through a surface ($\vec{F}.d\vec{s}$) to the behavior of vector field ($\vec{\nabla}.\vec{F}$) inside the volume.

3.2.1 Deduction of Gauss law with Gauss Divergence theorem:

In electrostatics the Gauss law is one of the fundamental law and frequently used. This law is a result of Gauss theorem in electric field.

Statement: The total electric flux through a closed surface is equal to $\frac{1}{\epsilon_o}$ times total charge enclosed inside the surface.

Mathematically: $\iint_S \vec{E}.d\vec{s} = \frac{1}{\epsilon_o}$ (total charge inside the surface)

$$\iint_S \vec{E}.d\vec{s} = \frac{1}{\epsilon_o}\ \sum_i q_i$$

Proof: Let us consider a charge q is situated at O, the origin of Cartesian coordinate system. Consider an imaginary surface called Gaussian surface around the charge q. The Gaussian surface may be of any shape but closed.

Consider a small surface ds on the Gaussian surface as shown is figure3.2. The distance (radial) of this surface is r from the origin and it subtends a solid angle $d\omega$ at the centre.

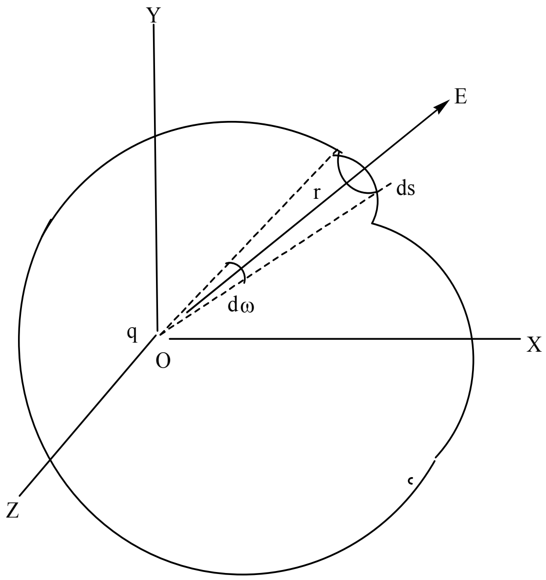


figure 3.2

The electric flux through this small surface ds is

$$d\emptyset = \vec{E}.d\vec{s}$$

The total electric flux through the whole surface

$$\emptyset = \iint_s \vec{E}\,.d\vec{s}$$

Now the electric field on the surface ds is given by.

$$E = \frac{1}{4\pi \in_0}\frac{q}{r^2}\hat{r}.d\vec{s}$$

where $\hat{r}$ is unit vector along the direction of $\vec{r}$. The flux $\emptyset$

$$\emptyset = \iint_s \frac{1}{4\pi \in_0}\frac{q}{r^2}\hat{r}.d\vec{s}$$

$$= \frac{1}{4\pi \in_0}\iint_s q.\frac{\hat{r}\,.\hat{n}\;d\vec{s}}{r^2}$$

$$where\ \hat{n}\ is\ unit\ vector\ perpendicualr\ to\ surface\ ds.$$

$$\emptyset = \frac{1}{4\pi \in_0}\iint_s \frac{q\;ds\;cos\theta}{r^2}$$

Where $\hat{r}.\hat{n} = cos\theta$. Now $\frac{ds\,cos\theta}{r^2}$ is solid angle subtended by surface ds and denoted by $d\omega$.

$$\emptyset = \frac{1}{4\pi \in_0} \iint_s q.d\omega = \frac{1}{4\pi \in_0}.q.4\pi = \frac{q}{\in_0}$$

Since total angle subtended by whole surface S at the centre is 4π.

Hence $\iint_s E.ds = \frac{1}{\in_0}\sum_i q_i$

In case the charge in the closed surface is distributed in the volume V with volume charge density ρ then the statement can be given as

$$\emptyset = \iint_s E.ds = \frac{1}{\in_0}\iiint_v \rho\, dV$$

3.2.2 Gauss law in differential form:

Gauss law in electrostatics is given as

$$\iint_s \vec{E}.d\vec{s} = \frac{1}{\in_0}\ (total\ charge\ inside\ surface\ s)$$

If ρ is volume charge density inside the volume and is enclosed by surface s then,

$$\iint_s \vec{E}.d\vec{s} = \frac{1}{\in_0}\iiint_v \rho dV$$

$$\iint_s \vec{E}.d\vec{s} = \iiint_v div\,\vec{E}\,dv$$

Applying Gauss divergence theorem

$$\iiint_v div\,\vec{E}\ \ dV = \frac{1}{\in_0}\iiint_v \rho dV$$

$$\iiint_v \left(div\,\vec{E} - \frac{\rho}{\in_0}\right) dV = 0$$

$$div\,\vec{E} - \frac{\rho}{\in_0} = 0$$

$$div\,\vec{E} = \frac{\rho}{\in_0}$$

This is called differential equation of Gauss law.

3.2.3: Poisson's equation and Laplace equation:

If we consider $\vec{E}$ as electric field and $\emptyset$ as electric potential then the electric field can be given as

$$\vec{E} = -\nabla\emptyset$$

Now using differential form of Gauss law

$$div\ (-\nabla\emptyset) = \frac{\rho}{\epsilon_0}$$

$$\nabla(-\nabla\emptyset) = \frac{\rho}{\epsilon_0}$$

$$\nabla^2\emptyset = -\frac{\rho}{\epsilon_0}$$

This is called Poisson's equation. Poisson's equation is basically second order differential equation and operator ∇^2 is an operator defined as

$$\nabla^2 = \nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

This is called Laplacian operator.

If there is no charge inside the volume i.e. ρ=0, then above equation becomes

$$\nabla^2\emptyset = 0$$

This is called Laplace equation.

Example3.1: If $\vec{r}$ is position vector of any point on the surface s whose volume is V, find $\iint_s \vec{r}.$ d$\vec{s}$.

Solution:

$$\iint_s \vec{r}\,.\overrightarrow{ds} = \iiint_v div\,\vec{r}\,dV$$

$$= \iiint_v \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(ix + jy + kz)\,dV$$

$$= \iiint_v \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right)\,dV$$

$$= \iiint_v 3\,dV = 3V.$$

Example3.2:

Using Gauss divergence theorem find out $\iint_s \vec{A} \, . d\vec{s}$ where, $A = x^3 i + y^3 j + z^3 k$ and s is a surface of a sphere defined by $x^2 + y^2 + z^2 = a^2$.

Solution:

$$\iint_s \vec{A}. d\vec{s} = \iiint_v \nabla. \vec{A} \, dV$$

$$= \iiint_v \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right). (x^3 i + y^3 j + z^3 k) \, dV$$

$$= \iiint_v (3x^2 + 3y^2 + 3z^2) \, dV$$

$$= 3 \iiint_v (x^2 + y^2 + z^2) \, dV$$

$$= 3 \iiint_v a^2 \, dV = = 3a^2 \iiint_v \, dV$$

$$= 3a^2 \left(\frac{4}{3}\pi a^3 \right) = \left(\frac{12}{3}\pi a^5 \right)$$

3.3 Green's Theorem for a Plane:

Statement: If $\emptyset_1(x, y)$ and $\emptyset_2(x, y)$ are two scalar functions which are continuous and have continuous derivatives $\frac{\partial \emptyset_1}{\partial y}$ *and* $\frac{\partial \emptyset_2}{\partial x}$ over a region R bounded by simple closed curve c in x-y plane, then

$$\oint_c (\emptyset_1 dx + \emptyset_2 dy) = \iint_R \left(\frac{\partial \emptyset_2}{\partial x} - \frac{\partial \emptyset_1}{\partial y}\right) dxdy$$

Proof: Let us consider a close path ABCD denoted by curve c, and curve c divided into two parts curve c_1(ABC) and $c_2(CDA)$ as shown in figure 3.3.

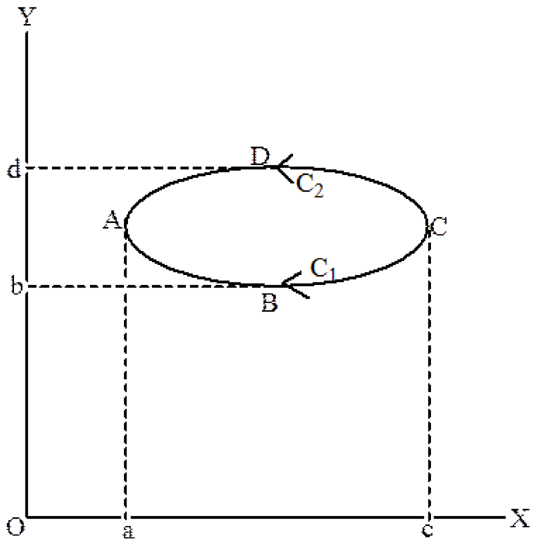


Figure 3.4

The equation of curve c_1 is

$$y = y_1(x)$$

The equation of curve c_2 is

$$y = y_2(x)$$

First we calculate the value of

$$\iint_R \frac{\partial \emptyset_1}{\partial y} dxdy = \int_{x=a}^{x=c} \left[\int_{y=y_{1(x)}}^{y=y_2(x)} \frac{\partial \emptyset_1}{\partial y} dy\right] dx$$

$$= \int_a^c [\emptyset_1(x,y)]_{y_{1(x)}}^{y_2(x)} dx$$

$$= \int_a^c [\emptyset_1(x,y_2) - \emptyset_1(x,y_1)]\, dx$$

$$= -\int_c^a \emptyset_1(x,y_2)\, dx - \int_a^c \emptyset_1(x,y_1) dx$$

$$= -[\int_c^a \emptyset_1(x,y_2) dx + \int_a^c \emptyset_1(x,y_1) dx]$$

$$= -\left[\int_{c_2} \emptyset_1(x,y) dx + \int_{c_1} \emptyset_1(x,y) dx\right]$$

$$= -\oint_c \emptyset_1\,(x,y) dx$$

Thus $\oint_c \emptyset_1(x,y)dx = -\iint_R \frac{\partial \emptyset_1}{\partial y} dxdy$

Similarly it can be proved that

$\oint_c \emptyset_2(x,y)dy = +\iint_R \frac{\partial \emptyset_2}{\partial x} dxdy$

Adding above two equations

$$\oint_c (\emptyset_1(x,y)dx + \emptyset_2(x,y)dy) = \iint_R \left(\frac{\partial \emptyset_2}{\partial x} - \frac{\partial \emptyset_1}{\partial y}\right) dxdy$$

$$\oint_c (\emptyset_1 dx + \emptyset_2 dy) = \iint_R \left(\frac{\partial \emptyset_2}{\partial x} - \frac{\partial \emptyset_1}{\partial y}\right) dxdy$$

This is Green's theorem for a plane.

Example3.3: A vector field $\vec{F}$ is given by $\vec{F} = siny\ i + x(1 + cosy)j$. Evaluate the line integral $\int_c \vec{F}.d\vec{r}$, where c is the circular path given by $x^2 + y^2 = a^2$.

Solution: The vector field F is given as
$\vec{F} = \sin y\ \ i + x(1 + cos\ y)j$

Taking line integral along the curve c

$$\int_c F.dr = \int_c [sin\ y\ i + x(1 + cos\ y)j].(idx + jdy)$$

$$= \int_c (\sin y\, dx + x(1 + \cos y)dy)$$

$$here\ \overline{r} = ix + jy \quad or \quad d\overline{r} = idx + jdy$$

Now take $sin\ y = \emptyset_1$ and $x(1 + cos\ y) = \emptyset_2$

On applying Green's theorem

$$\int_c (\emptyset_1 dx + \emptyset_2 dy) = \iint_R \left(\frac{\partial \emptyset_2}{\partial x} - \frac{\partial \emptyset_1}{\partial y}\right) dxdy$$

$$= \iint_R \left[\frac{\partial(x(1 + cos\ y))}{\partial x} - \frac{\partial\ sin\ y}{\partial y}\right] dxdy$$

$$= \iint_R [(1 + \cos y) - \cos\ y]dxdy$$

$$= \iint_R dxdy = \pi a^2$$

Since R is the region of circular path along xy plane given by $x^2 + y^2 = a^2$. Therefore the radius of circular path is a.

Example3.4: Applying Green's theorem evaluate

$$\int_c [(x^2 + 3xy)dx + (x^2 + y^2)dy]$$

Where c is a curve which form a square between the line $y = \pm 1 \; and \; x = \pm 1$

Solution: Given integral is

$\int_c [(x^2 + 3xy)dx + (x^2 + y^2)dy]$

Applying Green's theorem

$\oint_c (\emptyset_1 dx + \emptyset_2 dy) = \iint \left(\frac{\partial \emptyset_2}{\partial x} - \frac{\partial \emptyset_1}{\partial y}\right) dxdy$

$$= \int_{-1}^{1} \int_{-1}^{1} \left[\frac{\partial (x^2+y^2)}{\partial x} - \frac{\partial}{\partial y}(x^2 + 3xy)\right] dxdy$$

$$= \int_{-1}^{1} \int_{-1}^{1} [2x - 3x)] dxdy$$

$$= -\int_{-1}^{1} \int_{-1}^{1} xdxdy = -\int_{-1}^{1} xdx \int_{-1}^{1} dy$$

$$= -\left[\frac{x^2}{2}\right]_{-1}^{1} [y]_{-1}^{1} = -\frac{1}{2}(1^2 - 1^2)(1 + 1) = 0 \text{ Answer.}$$

3.4 Stoke's Theorem:

Stoke's theorem transforms the surface integral of the curl of a vector into line integral of that vector over the boundary C of that surface.

Statement: The surface integral of the curl of a vector taken over the surface **s** bounded by a curve c is equal to the line integral of the vector A along the closed curve c.

Mathematically:

$$\iint_s Curl\vec{A}.d\vec{s} = \oint_c \vec{A}.d\vec{r}$$

Since the curl A of a vector or vector function is along the normal to the surface, therefore the above statement may also be represented as

$$\iint_s curl\,\vec{A}.\hat{n}\,ds = \oint_c \vec{A}.d\vec{r}$$

Where $\hat{n}$ is a unit vector perpendicular to the surface ds. Unit vector $\hat{n}$ can be given as

$$\hat{n} = \cos\alpha\, i + cos\beta\, j + cos\gamma\, k$$

Proof: Let us consider a vector function A given as

$\vec{A}$=$A_x i + A_y j + A_z k$

And $\vec{r} = xi + yj + zk$

$d\vec{r} = idx + jdy + kdz$

Using the Stoke's theorem $\int_c \vec{A}.\overrightarrow{dr} = \iint_s curl\, A.\hat{n}\, ds$

$\int_c (A_x i + A_y j + A_z k).(idx + jdy + kdz)$

$$= \iint_s \left[\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times (A_x i + A_y j + A_z k)\right].(i\, cos\alpha + j\, cos\beta + k\, cos\gamma)ds$$

or

$\int_c (A_x dx + A_y dy + A_z dz) = \iint_s \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) i + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) j + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) k\right].(i\, cos\alpha + j\, cos\beta + k\, cos\gamma)\, ds$

or $\int_c (A_x dx + A_y dy + A_z dz) = \iint_s \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) cos\alpha + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) cos\beta + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) cos\gamma)\right] ds$

or

$$\int_c (A_x dx + A_y dy + A_z dz) = \iint_s \left[\left(\frac{\partial A_x}{\partial z} cos\beta - \frac{\partial A_x}{\partial y} cos\gamma\right) + \left(-\frac{\partial A_y}{\partial z} cos\alpha + \frac{\partial A_y}{\partial x} cos\gamma\right) + \left(\frac{\partial A_z}{\partial y} cos\alpha - \frac{\partial A_z}{\partial x} cos\gamma\right)\right] ds \quad (1)$$

Let us first prove the first term

$$\int_c A_x dx = \iint_s \left(\frac{\partial A_x}{\partial z} cos\beta - \frac{\partial A_x}{\partial y} cos\gamma\right) ds \quad (2)$$

Consider the A_x is function of (x, y, z) as $A_x(x, y, z)$ and $z = g(x, y)$ describes an equation of surface s, and ds is a small elementary part of this surface as shown in figure 3.4.

$\int_C A_x(x, y, z)dx = \int_c A_x(x, y, g(x, y))dx$

$$= \int_c [A_x + \frac{\partial A_x\,(x,y,g(x,y))}{\partial y} dy]\ \mathrm{dx}$$

By using Green's theorem

$$\int_C A_x(x,y,z)dx = \iint_s (\frac{\partial A_x}{\partial y} + \frac{\partial A_x}{\partial z}\frac{\partial g}{\partial y})\, dxdy \qquad (3)$$

The direction cosines of the normal to the surface s are given as

$$\frac{Cos\alpha}{-\frac{\partial g}{\partial x}} = \frac{Cos\beta}{-\frac{\partial g}{\partial y}} = \frac{Cos\gamma}{1}$$

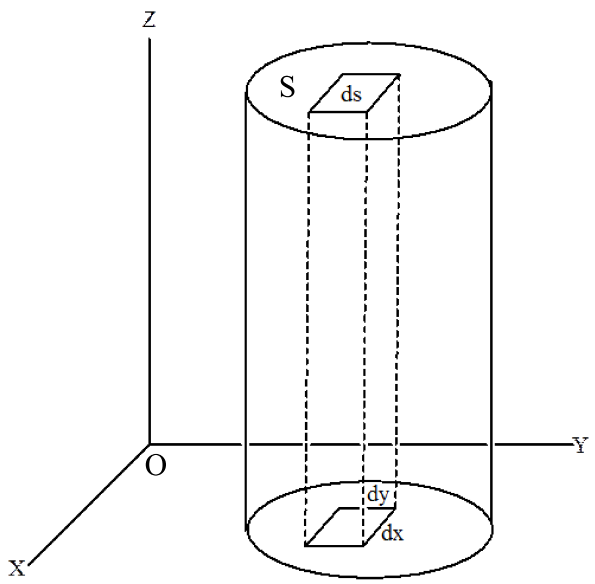


Figure 3.4

If the projection of ds on x-y plane is $ds\ cos\gamma$

Then $dxdy = ds\ cos\gamma$ or $ds = \frac{dxdy}{cos\gamma}$

Putting this value on equation (2)

$$\iint_s \left(\frac{\partial A_x}{\partial z} Cos\beta - \frac{\partial A_x}{\partial y} cos\gamma\right) ds = \iint_s (\frac{\partial A_x}{\partial z} Cos\beta - \frac{\partial A_x Cos\gamma}{\partial y}) \frac{dxdy}{cos\gamma}$$

$$= \iint_s \left(\frac{\partial A_x}{\partial z}\frac{Cos\beta}{Cos\gamma} - \frac{\partial A_x}{\partial y}\right) dxdy$$

$$= \iint_s \left[\frac{\partial A_x}{\partial z}\left(-\frac{\partial g}{\partial y}\right) - \frac{\partial A_x}{\partial y}\right] dxdy$$

$$= -\iint [\frac{\partial A_x}{\partial y} + \frac{\partial A_x}{\partial z}.\frac{\partial g}{\partial y}]dxdy$$

Putting the value of R.H.S. from equation (3)

$\iint_s(\frac{\partial A_x}{\partial z}Cos\beta - \frac{\partial A_x}{\partial y}cos\gamma)\ ds = \int_C A_x(x,y,z)dx$

or $\int_C A_x(x,y,z)dx = \iint_s(\frac{\partial A_x}{\partial z}Cos\beta - \frac{\partial A_x}{\partial y}cos\gamma)\ ds$ (4)

Similarly

$\int_c A_y dy = \iint_s(\frac{\partial A_y}{\partial x}Cos\gamma - \frac{\partial A_y}{\partial z}Cos\alpha)\, ds$ (5)

and $\int_c A_z dz = \iint_s(\frac{\partial A_z}{\partial y}Cos\gamma - \frac{\partial A_z}{\partial x}Cos\beta)ds$ (6)

On adding above equations (4), (5) and (6)

$\int_c\left(A_x dx + A_y dy + A_z dx\right) = \iint_s\left[\frac{\partial Ax}{\partial z}Cos\beta - \frac{\partial A_x}{\partial y}Cos\gamma + \frac{\partial A_y}{\partial x}cos\gamma - \frac{\partial A_y}{\partial z}Cos\alpha + \frac{\partial A_z}{\partial y}\cos\ r - \frac{\partial A_z}{\partial x}cos\beta\right] ds$

Or $\int_c \vec{A}.d\vec{r} = \iint_s Curl\vec{A}.d\vec{s}$

Hence Stoke’s theorem is proved.

Example3.5: Using Stoke’s theorem evaluate

$\int_c[(2x-y)dx - yz^2dy - y^2zdz]\ where\ c\ is\ the\ circle\ x^2+y^2 = 1$ Corresponding to the surface of a sphere of radius 1.

Solution:

The given integral

$$\int_c [(2x-y)dx - yz^2dy - y^2zdz]$$

$$= \int_c [(2x-y)i - yz^2j - y^2zk].(idx + jdy + kdz)$$

$$= \int_c \vec{A}.d\vec{r}$$

Where $\vec{A} = (2x-y)i - yz^2j - y^2zk$ and $d\vec{r} = idx + jdy + kdz$

Using stokes theorem $\int_c \vec{A}.d\vec{r} = \iint_s \nabla \times \vec{A}.\ d\vec{s}$

$$= \iint_s \nabla \times \mathrm{A}.\hat{\mathrm{n}}\ \mathrm{ds} \qquad (1)$$

where $\hat{\mathrm{n}}$ = unit vector perpendicualr to surface ds

$$\nabla \times \bar{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yZ^2 & -y^2z \end{vmatrix}$$

= i(-2zy+2yz)-j(0-0)+k(0+1)

=k

Putting this value in equation (1)

$$\int_c \vec{A}.d\vec{r} = \iint_s k.\hat{n}ds$$

Since ds is area of a circle described by $x^2 + y^2 = 1$ along xy plane, therefore direction of ds is along perpendicular to surface which is along z axis.

Thus $\iint_s k.\hat{n}ds = \iint_s ds = \iint\ dxdy = \pi$

Answer.

Example3.6: Verify Stoke's theorem for vector filed given by $\vec{F} = (3x-2y)i + x^2zj + y^2(z+1)k$ for a plane rectangular area with corners at (0,0),(1,0) (1,2) and (0,2) in x-y plane.

Solution: the given function is

$$\vec{F} = (3x-2y)i + x^2z\,j + y^2(z+1)k$$

Since the vector field is applying in an area which is described in x-y plane only, therefore z=0 and function becomes

$$\vec{F} = (3x-2y)i + y^2k \qquad (1)$$

According to Stoke's theorem

$$\int_c \vec{F}.\overrightarrow{dr} = \iint_s \nabla \times \vec{F}.\overrightarrow{ds} \qquad (2)$$

The line integral along the close path described by rectangle OADC as shown in figure 3.5 and can be given as

$$\int_C \vec{F}.\overrightarrow{dr} = \int_{C_1} \vec{F}.\overrightarrow{dr} + \int_{C_2} \vec{F}.\overrightarrow{dr} + \int_{C_3} \vec{F}.\overrightarrow{dr} + \int_{C_4} \overrightarrow{F}.\overrightarrow{dr}$$

Where C_1, C_2, C_3, C_4 are components of curve C.

$$\int_C F.dr = \int_0^1 [(3x-2y)i + y^2k].idx + \int_0^2 [(3x-2y)i + y^2k].jdy$$

$$+\int_1^0 [(3x-2y)i + y^2k].idx + \int_2^0 [(3x-2y)i + y^2k].jdy$$

$$= \int_0^1 3xdx + \int_0^2 0\, dy + \int_1^0 (3x-4)dx + \int_2^0 0.dy$$

$$= \frac{3}{2} + 0 + \frac{5}{2} + 0 = 4$$

Answer.

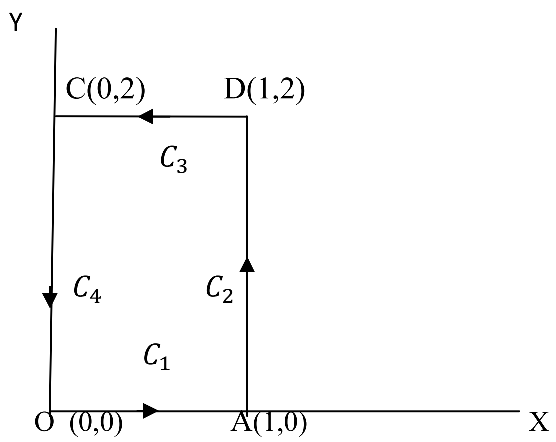


Figure 3.5

The L.H.S of equation (2) become 4 for given field. Now we calculate the R.H.S of equation (2)

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x-2y & 0 & y^2 \end{vmatrix}$$

$$= (2y\, i + 0 + 2k) = 2y\, i + 2k$$

Now $\iint_s \nabla \times F.ds = \iint_s (2yi + 2k).\hat{n}\ dxdy$

$$= \iint_s 2dxdy = 2\iint_s dxdy$$

=2. Area of rectangle =2.2=4

On comparing equation (3) and (4) the Stoke's theorem has been verified.

3.5 Summary:

1. Gauss divergence theorem transforms surface integral into volume integral and vice-versa. The theorem states that the surface integral of a vector filed $\vec{F}$ over a closed surface s is equal to the volume integral of divergence of $\vec{F}$ taken over the volume enclosed by surface s.

$$\iint_s \vec{F}.\overrightarrow{ds} = \iiint_v div\, \vec{F}\, dv$$

2. Gauss law is a result of Gauss theorem in electric field. According to this law the total electric flux through a closed surface is equal to $\frac{1}{\epsilon_o}$ times total charge enclosed inside the surface.

$$\iint_s E.ds = \frac{1}{\epsilon_o}\text{ (total charge inside the surface)}$$

3. Gauss law in differential form:

$$div\, E = \frac{\rho}{\epsilon_0}$$

4. Poisson's equation and Laplace equation:

$$\nabla^2\emptyset = -\frac{\rho}{\epsilon_0}$$

This is called Poisson's equation. Poisson's equation is basically second order differential equation and operator ∇^2 is an operator called Laplacian operator and defined as

$$\nabla^2 = \nabla.\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

If there is no charge inside the volume i.e. ρ=0, then above equation becomes Laplace equation

$$\nabla^2\emptyset = 0$$

5. Green's Theorem for a Plane: If $\emptyset_1(x, y)$ and$\emptyset_2(x, y)$ are two scalar functions which are continuous and have continuous derivatives $\frac{\partial\emptyset_1}{\partial y}$ *and* $\frac{\partial\emptyset_2}{\partial x}$ over a region R bounded by simple closed curve c in x-y plane, then

$$\oint_c(\emptyset_1 dx + \emptyset_2 dy) = \iint_R \left(\frac{\partial\emptyset_2}{\partial x} - \frac{\partial\emptyset_1}{\partial y}\right)dxdy$$

6. Stoke's Theorem: Stoke's theorem transforms the surface integral of the curl of a vector into line integral of that vector over the boundary C of that surface. According to this theorem the surface integral of the curl of a vector taken over the surface **s** bounded by a curve c is equal to the line integral of the vector A along the closed curve c.

$\iint_s curl\, \vec{A}.\overrightarrow{ds} = \oint_c \vec{A}.\overrightarrow{ds}$

3.6 Glossary

Transformation- conversion

Flux – scalar product of a field vector and area

divergence- rate at which density exits in a given region of space. (flux density)

Curl- describes the rotation of vector field.

3.7 Self assessment questions:

1. Express the divergence theorem in words and mathematical form.

2. Calculate the divergence of vector function $\boldsymbol{A} = xy\, i + yz\, j + zxk$

3. Prove Gauss theorem in the form

$$\iint_s d\vec{S}.\vec{F} = \iiint_V (\nabla.\vec{F}\,)dV$$

3. Using Gauss divergence theorem evaluate

$\iint_s x\, dxdy + y\, dzdx + z\, dxdy$ Where S is surface of a sphere given by $x^2 + y^2 + z^2 = 1$.

4. A vector is defined as $\vec{r} = xi + yj + zk$. Then show that $\oint \vec{r}.\overrightarrow{dr} = 0$

5. Explain the term flux of a vector field.

3.8 Reference Books:

1. Mechanics – D.S. Mathur, S Chand, Delhi
2. Concept of Physics- H C Verma, Bharti Bhawan, Patna
3. Physics Part-I, Robert Resnick and David Halliday, Wiley Eastern Ltd

3.9 Suggested readings:

1. Modern Physics, Beiser, Tata McGraw Hill

2. Fundamental University Physics-I, M. Alonslo and E Finn, Addition-Wesley Publication
3. Berkeley Physics Course Vol I, Mechanics, C Kittel et al, McGraw- Hill Company

3.10 Terminal questions:

3.10.1 Short answer type questions

1. Explain the physical significance of Gauss's divergence theorem.

2. If F is a scalar function which is solution of Laplace equation $\nabla^2 F = 0$ in a volume V bounded by the piecewise smooth surface S, then apply the Gauss theorem and show that

$$\iint_s \hat{n}.\nabla F \; dS = 0$$

3. Verify Green's theorem in a plane for $[(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where C is boundary of a region defined by $x = 0, y = 0, x + y = 1$

4. Prove that $\hat{n}.dS = 0$ and $\iint_s (\nabla \times \vec{F}\,) . \overrightarrow{dS} = 0$

5. If the line integral of a vector $\vec{A}$ around a closed curve is equal to the surface integral of the vector $\vec{B}$ taken over the surface bounded by the given closed curve then show that

$\vec{B} = curl\,\vec{A}$.

3.10.2 Essay type questions

1. State and proof Gauss's divergence theorem.

2. State and prove Stoke's theorem in vector analysis.

3. State and prove Green's theorem in a plane.

4. Verify Green's theorem in a plane for $\oint_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where c is the boundary defined by $y = x^{1/2}; y = x^2$.

UNIT 4: NEWTON'S LAWS OF MOTION AND CONSERVATION PRINCIPLES

STRUCTURE:

4.1 INTRODUCTION

An object is said to be in motion if its position changes with respect to its surroundings in a given time while on the other hand, if the position of the object does not change with respect to its surroundings, it is said to be at rest. A motorbike speeding on road, a bird flying through air, a ship sailing on water, the graceful movements of a dancer are the examples of objects in motion while on the other hand, a pen lying on the table is at rest because its position with respect to the table does not change with time.

To study the motion of an object, you have to study the change in the position of the object with respect to its surroundings. In space, the position of an object is specified by the three coordinates x, y and z. The position of the object changes due to change in one or two or all the three coordinates. The motion of an object is said to be one-dimensional when one of the three coordinates specifying the position of the object changes with time. The motion of a bus on the road, the motion of a train on railway track or an object falling freely under gravity are examples of one-dimensional motion. The motion of an object is said to be two-dimensional when two of the three coordinates specifying the position of the object change with time. Among the well-known examples of two-dimensional motion that you have studied are circular motion and projectile motion. However, your study was limited to the motion along a straight line (one-dimensional motion) and in a two-dimensional plane (two-dimensional motion). But you know that our world is three-dimensional in space. Therefore, we shall begin by studying motion in three dimensions. The motion of an object is said to be three-dimensional when all the three coordinates specifying the position of the object change with time. The motion of a flying kite, gas molecules or the motion of bird in the sky are some examples of three-dimensional motion.

We shall first understand what we mean when we say that an object is moving. We shall learn how to describe the motion of a particle in terms of displacement, velocity and acceleration. In this unit, we shall also study the factors affecting the motion. For this, we shall study Newton's laws of motion and apply them to a variety of situations. We shall use the familiar concept of linear momentum to study the motion of systems having more than one particle and establish the principle of conservation of linear momentum.

4.2 OBJECTIVES

After studying this unit, you should be able to-

- define motion
- apply Newton's laws of motion
- solve problems using Newton's laws of motion
- apply the law of conservation of linear momentum

- apply the impulse of force

4.3 WHAT IS MOTION?

Can you imagine what your life would be like if you were confined to some place, unable to move from one position to another as the time passed? This sentence possess the answer to the question. What is motion? If an object occupies different positions at different instants of time, then we say that it is moving or it is in motion. Thus the study of motion deals with questions- where? And when? Let us recall some definitions relating to motion.

4.3.1 Distance and Displacement

The position of a moving object goes on changing with respect to time. The length of the actual path covered by a body in a time-interval is called 'distance' while the difference between the final and the initial positions of an object is known as 'displacement'. We know that the position of an object is always expressed with respect to some reference point. If the initial position of an object with respect to a reference point is x_1 and after some time it becomes x_2, then the magnitude of the displacement of the object is $x_2 - x_1$.

In your school science courses, you have studied an important difference between 'distance' and 'displacement'.

4.3.2 Speed

The distance travelled by an object in unit interval of time is called the 'speed' of the object i.e.

$$\text{speed} = \frac{\text{distance}}{\text{time}-\text{interval}} \qquad \text{.....(1)}$$

The speed is represented by 'v' and it has unit meter/second. It is a scalar quantity.

4.3.3 Velocity

The displacement of an object in a particular direction in unit interval of time is called the 'velocity' of the object i.e.

$$\text{velocity} = \frac{\text{displacement}}{\text{time}-\text{interval}} \qquad \text{.....(2)}$$

The velocity is also represented by $\vec{v}$ and its unit is the same as that of speed i.e. meter/second. You know that velocity is a vector quantity.

Let us suppose that an object is moving along a straight line and with respect to some reference point, its position is x_1 at time t_1 and becomes x_2 at time t_2. It means that in time-interval $(t_2- t_1)$, the displacement of the object is $(x_2- x_1)$. Hence, the average velocity of the object during this time-interval would be-

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} \qquad .....(3)$$

For expressing the difference in a quantity, we use the symbol Δ (delta). Therefore, we can write the average velocity of the object as-

$$\bar{v} = \frac{\Delta \vec{X}}{\Delta t} \qquad .....(4)$$

If we go on decreasing the time-interval Δt and when Δt becomes infinitesimally small ($\Delta t \to 0$), then from the above formula, we shall be knowing the velocity of the object at a particular instant of time. This velocity is called the 'instantaneous velocity' of the object and is given by

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta x}}{\Delta t} = \frac{\overrightarrow{dx}}{dt} \qquad .....(5)$$

4.3.4 Acceleration

You should know that if the velocity of a moving object is changing then its motion is known as 'accelerated motion'. It is obvious that change in velocity means the change in magnitude (i.e. speed) or in direction or in both. Thus, the time-rate of change of velocity of an object is called the 'acceleration' of that object, i.e.

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{time-interval}} \qquad .....(6)$$

Acceleration is generally represented by 'a' and it has unit meter/second^2. It is also a vector quantity.

Let us suppose that the velocity of a moving object at time t_1 is v_1 and at time t_2, it becomes v_2. It means that in the time-interval (t_2- t_1), the change in the velocity of the object is (v_2- v_1). Therefore, the average acceleration of the object in time-interval (t_2- t_1) is

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\overrightarrow{\Delta v}}{\Delta t} \qquad .....(7)$$

If the time-interval Δt is infinitesimally small (i.e. $\Delta t \to 0$), then at a particular time, the instantaneous acceleration is given by

$$a = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta v}}{\Delta t} = \frac{\overrightarrow{dv}}{dt} \qquad .....(8)$$

If the velocity of an object undergoes equal changes in equal time-intervals, then its acceleration is called 'uniform'

Example 1: The displacement versus time equation of a particle falling freely from rest is given by x= (2.9 ms^{-2}) t^2, where x is in meters, t in seconds. Calculate the average velocity of the particle between t_1= 2 sec and t_2 = 3 sec.

Solution: Here x= (2.9 ms^{-2}) t^2

At time t_1 = 2 sec, x_1= 2.9 x $(2)^2$ = 11.6 m and at time t_2= 3 sec, x_2 = 2.9 x $(3)^2$ = 26.1 m

Average velocity $\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{26.1 - 11.6}{3 - 2}$ = 14.5 m/s

Self Assessment Question (SAQ) 1: A particle moves along the x-axis in such a way that its coordinate (x) varies with time (t) according to the expression x = 2- 6t+ $8t^2$ meter. Find the initial velocity of the particle.

Self Assessment Question (SAQ) 2: Can a body have zero velocity and finite acceleration? Give example.

Self Assessment Question (SAQ) 3: If the displacement of a body is proportional to square of time, state whether the body is moving with uniform velocity or uniform acceleration.

Self Assessment Question (SAQ) 4: Choose the correct option-

The distance covered by a particle as a function of time t is given by x= $5t^3$ + 6 t^2 -5. The acceleration of the particle-

(i) remains constant (ii) increases with time (iii) decreases with time (iv) first increases and then decreases with time

4.4 CAUSES OF MOTION

What makes things move? Let us understand this question. The answer of this question was suggested by the great physicist Aristotle, way back in the fourth century B.C. Most of the people believed in his answer that a force which is described as push or pull, was needed to keep something moving. And the motion ceased when the force was removed. This idea made a lot of common sense. But these ideas were first critically examined by Galileo who performed a series of experiments to show that no cause or force is required to maintain the motion of an object.

Let us try to understand some common examples. From our daily experience, we know that the motion of a body is a direct result of its interactions with the other bodies around it which form its environment when a cricketer hits a ball, his or her bat interacts with the ball and modifies its motion. The motion of a freely falling body or of a projectile is the result of its interaction with earth. When an ox stopped pulling an ox-cart, the cart quickly comes to a stop.

An interaction is quantitatively expressed in terms of a concept called ‘force’- a push or a pull. When we push or pull a body, we are said to exert a force on it. Earth pulls all bodies towards its centre and is said to exert a force (gravitational) on them. A locomotive exerts a force on the train, it is either pulling or pushing. In this way every force exerted on a body is associated with some other body in the environment.

We should also remember that it is not always that an application of force will cause motion or change motion. For example, we may push a wall i.e. there is an interaction between us and the wall and hence there is a force, but the wall may not move at all.

Thus, force may be described as push or pull, resulting from the interaction between bodies which produces or tends to produce motion or change in motion. The analysis of the relation between force and motion of a body is based on Newton's laws of motion. We will now discuss these laws.

4.4.1 Newton's Laws of Motion

Galileo concluded that any object in motion, if not obstructed, will continue to move with a constant speed along a horizontal line. Therefore, there would be no change in the motion of an object, unless an external agent acted on it to cause the change. That was Galileo's version of inertia. Inertia resists changes, not only from the state of rest, but also from motion with a constant speed along a straight line. Therefore, the interest shifted from the causes of motion to the causes for changes in motion. Galileo's version of inertia was formalized by Newton in a form that has come to be known as Newton's first law of motion.

4.4.2 Newton's First Law

This law states –"**If a body is at rest then it will remain at rest or if it is moving along a straight line with a uniform speed then it will continue to move as such unless an external force is applied on it to change its present state**". This property of bodies showing a reluctance to change their present state is called "inertia". Hence, Newton's first law is also known as the "law of inertia" This law is also called "Galileo law". Newton's first law makes no distinction between a body at rest and one moving with a constant velocity. Both states are 'natural' when no net external force or interaction acts on the body.

4.4.3 Newton's Second Law

Newton's second law tells us what happens to the state of rest or of uniform motion of a body when a net external forces acts on the body i.e. when the body interacts with other surroundings bodies. This law states –"**The time-rate of change of linear momentum of a particle is directly proportional to the force applied on the particle and it takes place in the direction of the force**".

Mathematically,

$$\frac{d\vec{p}}{dt} \propto \vec{F}$$

or $\frac{d\vec{p}}{dt} = k\,\vec{F}$, where $\vec{F}$ is the applied force and k is a constant of proportionality. The differential operator $\frac{d}{dt}$ indicates the time-rate of change.

In MKS or SI system, k = 1. Therefore, $\frac{d\vec{p}}{dt} = \vec{F}$

or simply, $\vec{F} = \frac{d\vec{p}}{dt}$ …..(9)

But linear momentum of the particle $\vec{p} = m\vec{v}$, where m is the mass of the particle and $\vec{v}$ is the velocity of the particle.

Therefore, $\vec{F} = \frac{d(m\vec{v})}{dt} = m\,\frac{d\vec{v}}{dt} = m\vec{a}$ (since $\vec{a} = \frac{d\vec{v}}{dt}$ = acceleration of the particle)

Thus, Newton's second law can be written as $\vec{F} = m\vec{a}$ …..(10)

In scalar form, it can be written as F= ma …..(11)

Thus, force is equal to mass times acceleration, if the mass is constant. The force has the same direction as the acceleration. This is an alternative statement of the second law. Newton's second law is also known as 'Law of change in momentum'.

If the position vector of a particle is $\vec{r}$ at a time t then its velocity $\vec{v}$ can be expressed as-

$\vec{v} = \frac{\overrightarrow{dr}}{dt}$ …..(12)

and its acceleration $\vec{a} = \frac{\overrightarrow{dv}}{dt} = \frac{\overrightarrow{d^2r}}{dt^2}$ …..(13)

Therefore, Newton's second law can be written as-

$\vec{F} = m\frac{\overrightarrow{d^2r}}{dt^2}$ …..(14)

If the unit of mass m is kg and the unit of acceleration a is meter/second^2, then the unit of force F is called 'newton'.

If m=1 kg and a= 1 meter/second^2

then magnitude of force F = ma= 1 x 1 = 1 Newton

i.e. 1 newton force is the force which produces an acceleration of 1 meter/second^2 in a body of mass 1 kg.

4.4.4 Newton's Third Law

A force acting on a body arises as a result of its interaction with another body surrounding it. Thus, any single force is only one feature of a mutual interaction between two bodies. We find that whenever one body exerts a force on a second body, the second body always exerts on the first a force which is equal in magnitude but opposite in direction and has the same line of action. A single isolated force is therefore an impossibility.

The two forces involved in every interaction between the bodies are called an 'action' and a 'reaction'. Either force may be considered the 'action' and the other the 'reaction'. This fact is made clear in Newton's third law of motion. This law states-"**To every action there is always an equal and opposite reaction**".

Here the words 'action' and 'reaction' mean forces as defined by the first and second laws.

If a body A exerts a force $\overrightarrow{F_{AB}}$ on a body B, then the body B in turn exerts a force $\overrightarrow{F_{BA}}$ on A, such that

$\overrightarrow{F_{AB}} = -\overrightarrow{F_{BA}}$

So, we have $\overrightarrow{F_{AB}} + \overrightarrow{F_{BA}} = 0$ …..(15)

Notice that Newton's third law deals with two forces, each acting on a different body. This law is also known as 'Law of action-reaction'.

There are two important points regarding Newton's third law. Firstly, we cannot say that this particular force is action and the other one is reaction. Any one may be action and the other reaction. Secondly, action and reaction act on different bodies.

Out of three laws, Newton's second law is most general as first and third law may be derived from second law.

Example 2: A ship of mass 4 x 10^7 kg initially at rest is pulled by a force of 8 x 10^4 Newton through a distance of 4 meter. Assuming that the resistance due to water is negligible, calculate the speed of the ship.

Solution: Given mass of the ship m= 4 x 10^7 kg, Force applied F= 8 x 10^4 Newton

Using F= ma, the acceleration of the ship a= $\frac{F}{m} = \frac{8 \text{ x } 104}{4 \text{ x } 107} = 2 \times 10^{-3}$ m/sec^2

Distance s = 4 m, initial speed of the ship u= 0

Using equation of motion $v^2 = u^2 + 2as$

$$= (0)^2 + 2 \text{ x } 2 \text{ x } 10^{-3} \text{ x } 4 = 16 \text{ x } 10^{-3}$$

or $v = 0.1265$ m/sec

i.e. speed of the ship = 0.1265 m/sec

Example 3: A satellite in a force-free space sweeps stationary interplanetary dust at a rate $\frac{dm}{dt} = \alpha v$, calculate the acceleration of the satellite.

Solution: Given $\frac{dm}{dt} = \alpha v$

Using Newton's second law of motion $F = \frac{dp}{dt}$

$$= \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

Since force F= 0 (for force-free space), therefore $0 = m\frac{dv}{dt} + v\frac{dm}{dt}$

$= ma + v\,(\alpha v)$ [since $\frac{dv}{dt} = a$ and putting for $\frac{dm}{dt}$]

$= ma + \alpha v^2$

or $a = -\frac{\alpha v^2}{m}$

Thus, the acceleration of the satellite = $-\frac{\alpha v^2}{m}$

Self Assessment Question (SAQ) 5: When a player kicks a football, the football and the player experience forces of the same magnitude but in opposite directions according to Newton's third law. The football moves but the player does not move, Why ?

Self Assessment Question (SAQ) 6: A force produces an acceleration of 18 m/sec^2 in a body of mass 0.5 kg and an acceleration of 6 m/sec^2 in another body. If both the bodies are fastened together then how much acceleration will be produced by this force ?

Self Assessment Question (SAQ) 7: A person sitting in a bus moving with constant velocity along a straight line throws a ball vertically upward. Will the ball return to the hands of the person ? Why?

Self Assessment Question (SAQ) 8: According to Newton's third law, every force is accompanied by an equal and opposite force. How can a movement ever take place?

Self Assessment Question (SAQ) 9: Choose the correct option-

(a) When a constant force is applied to a body, it moves with uniform-

(i) acceleration (ii) velocity (iii) speed (iv) momentum

(b) Inertia is the property by virtue of which the body is-

(i) unable to change by itself the state of rest only

(ii) unable to change by itself the state of uniform linear motion only

(iii) unable to change by itself the direction of motion only

(iv) unable to change by itself the state of rest and of uniform linear motion

Self Assessment Question (SAQ) 10: Fill in the blank-

(i) Newton's first law of motion gives the concept of ……….
(ii) Newton's second law gives a measure of ……………..

4.5 WEIGHT AND MASS

The weight of a body is simply the gravitational force exerted by earth on the body. It is a vector quantity whose direction is the direction of the gravitational force i.e. towards the centre of the earth.

Let us understand the concept of weight in a better way. When a body of mass m falls freely, its acceleration is $\vec{g}$ and the force acting on it is its weight $\vec{w}$. Thus, by using Newton's second law, $\vec{F} = m\vec{a}$, we get

$$\vec{w} = m\vec{g} \qquad \text{.....(16)}$$

Since both weight $\vec{w}$ and acceleration due to gravity $\vec{g}$ are directed towards the centre of the earth, we can write $w = mg$ …..(17)

The mass m of a body is an intrinsic property of the body while the weight of a body is different in different localities because acceleration due to gravity g varies from point to point on the earth. The unit of mass is kg while that of weight is kg m/sec^2 or Newton.

4.6 APPLICATIONS OF NEWTON'S LAWS OF MOTION

Newton's laws of motion give us the means to understand most aspects of motion. Let us now apply them to a variety of physical situation involving objects in motion. To apply Newton's laws, we must identify the body whose motion interests us. Then we should identify all the forces acting on the body, draw them in the figure and find the net force acting on the body. Newton's second law can then be used to determine the body's acceleration. Let us use this basic method to solve a few examples.

4.6.1 Projectile Motion

The motion of a bullet fired by a gun and that of a ball thrown by a fieldsman to another are the examples of projectile motion. Let us consider such a particle of mass m. It is thrown from a point O with an initial velocity $\overrightarrow{v_0}$ along OP making an angle θ with the horizontal (Figure 1). Let the particle be at a point Q ($\overrightarrow{OQ} = \vec{r}$) at time t. Neglecting air resistance, let us determine the particle's path. The only force acting on the particle is the weight m$\vec{g}$ of the particle which is constant throughout the motion.

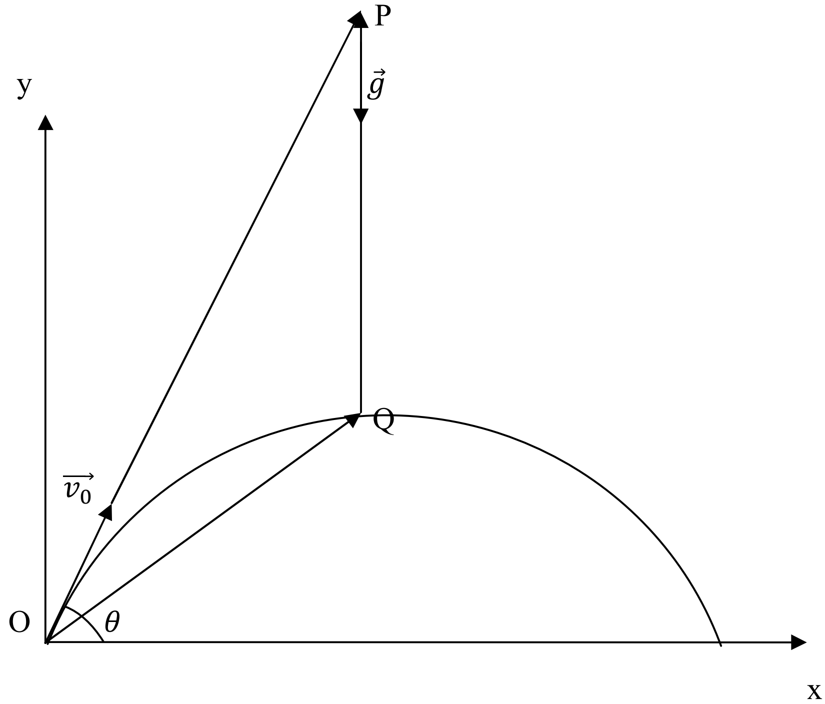


Figure 1

Let us determine the path of the particle. We know by Newton's second law

$\vec{F} = m\frac{\overrightarrow{d^2r}}{dt^2}$ …..(i)

But here $\vec{F} = m\vec{g}$

Therefore, $m\frac{\overrightarrow{d^2r}}{dt^2} = m\vec{g}$ or $\frac{\overrightarrow{d^2r}}{dt^2} = \vec{g}$

or $\frac{d}{dt}(\frac{\overrightarrow{dr}}{dt}) = \vec{g}$ …..(ii)

Integrating with respect to t, we get-

$\frac{\overrightarrow{dr}}{dt} = \vec{g}$t + A …..(iii)
where A is the constant of integration.
Applying initial condition, at t= 0, $\frac{\overrightarrow{dr}}{dt} = \overrightarrow{v_0}$, therefore, from equation (iii)
$\overrightarrow{v_0} = \vec{g}$ (0) + A
or A= $\overrightarrow{v_0}$
Putting for A in equation (iii), we get,
$\frac{\overrightarrow{dr}}{dt} = \vec{g}$t + $\overrightarrow{v_0}$ …..(iv)
Integrating equation (iv) with respect to time t, we get-
$\vec{r} = \overrightarrow{v_0}$ t + $\frac{1}{2}\,\vec{g}t^2$ + B …..(v)
Where B is a constant of integration.
Applying initial condition, at t = 0, $\vec{r}$ = 0, we get from equation (v)-
0 = $\overrightarrow{v_0}(0) + \frac{1}{2}\,\vec{g}(0)^2$ + B
or B = 0
Putting for B in equation (v), we get-
$\vec{r} = \overrightarrow{v_0}$ t + $\frac{1}{2}\,\vec{g}t^2$ …..(vi)

We have applied two initial conditions : $\frac{\overrightarrow{dr}}{dt} = \overrightarrow{v_0}$ and $\vec{r}$ = 0 at t = 0. Since $\overrightarrow{v_0}$ is along OP and t is scalar, we understand that $\overrightarrow{v_0}$ t is along OP. Again acceleration due to gravity $\vec{g}$ is directed vertically downwards and $\frac{1}{2}\,t^2$ is a scalar, therefore $\frac{1}{2}\,\vec{g}t^2$ is directed vertically downwards i.e. along PQ (Figure 1).
We use the law of vector addition-
$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$ …..(vii)
In this way, we get the location of the particle. As time advances OP is lengthened and therefore, is PQ, and we get the location of the particle by adding $\overrightarrow{OP}$ and $\overrightarrow{PQ}$.

4.6.2 Friction

A heavy block is kept on a horizontal rough floor. You apply a force to pull it but it still does not move. Is it a contradiction of Newton's laws? Let us discuss the motion of the block.

The block is acted upon by two forces-

(i) its weight mg acting vertically downward at its centre of gravity and
(ii) the reactionary-force P exerted on it by the floor directed vertically upward and passes through its centre of gravity.

Since the block is in equilibrium, P= mg. In the figure 2, the lines of action of mg and P are shown slightly displaced for clarity.

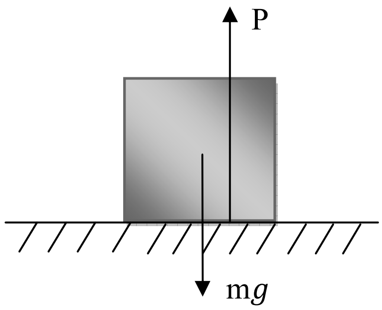
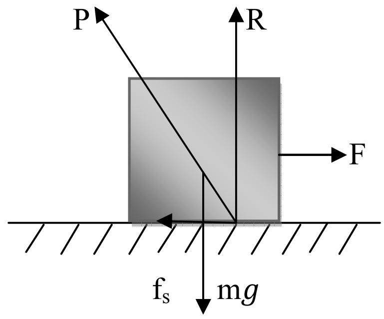


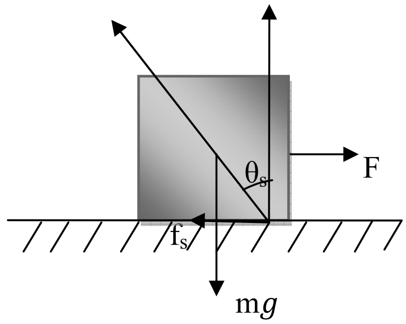
Figure 2



$f_s < \mu_s R$

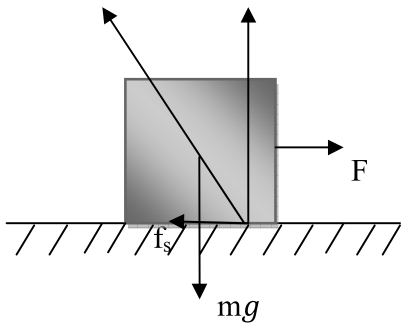
Figure 3

When we apply a small horizontal force F, say towards right (Figure 3), the block does not move. The force P exerted on the block by the floor is now so inclined towards left that P, mg and F may form a closed triangle (since block is still in equilibrium). The force P can be resolved into two components; parallel and perpendicular to the contact-surfaces. The component parallel to the contact surfaces is called the 'force of static friction' f_s which balances the applied force F (F= f_s). The component perpendicular to the contact surfaces is the 'normal reaction' R exerted on the block which balances the weight mg of the block (R= mg).



$f_s = \mu_s R$

Figure 4



$f_k = \mu_k R$

Figure 5

Now, if the applied force F is slightly increased, the block does not still begin to move. This means that the force P is further inclined towards left so that the force of static friction f_s also increases to become equal to the new value of F. Thus, as the applied force F is increased, the force of static friction f_s also increases, but after a certain limit, f_s cannot increase any more. At this moment the block is just to move (Figure 4). This maximum value of the static frictional force f_s is called 'limiting frictional force'(it is equal to the smallest force required to start motion). Now, as the applied force is further increased, the block begins to move.

The limiting (maximum) static frictional force depends upon the nature of the surfaces in contact. It does not depend upon the size or area of the surfaces. For the given surfaces, the limiting frictional force f_s is directly proportional to the normal reaction R i.e.

$f_s \propto R$

or $f_s = \mu_s R$ (for limiting frictional force) …..(i)

where the constant of proportionality μ_s is called the 'coefficient of static friction'. The above formula holds only when f_s has its maximum (limiting) value (Figure 4). Before this stage, $f_s < \mu_s R$ (Figure 3). Hence, usually, $f_s \leq \mu_s R$.

If the direction of the applied force is reversed, the direction of f_s also reverses, while the direction of R remains unchanged. in actual fact, f_s is always opposite to F.

Once the motion starts, the frictional force acting between the surfaces decreases, so that a smaller force F is required to maintain uniform motion (Figure 5). The force acting between the surfaces in relative motion is called the 'dynamic frictional force' f_k which is less than the limiting force of static friction f_s. You know from daily experience that a lesser force is required to maintain the motion of a body than the force required to start the body from rest.

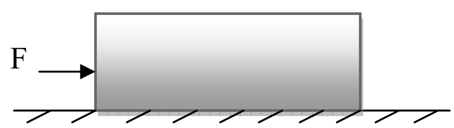
Thus, when the block is in uniform motion, the force of dynamic (or kinetic) friction is

$f_k = \mu_k R$ …..(ii)

where μ_k is the coefficient of dynamic (or kinetic) friction and its value is less than μ_s.

Let us see some examples based on these applications.

Example 4: A block of mass 2 kg is placed on a rough floor. The coefficient of static friction is 0.4. A force F of magnitude 2.5 N is applied on the block, as shown. Calculate the force of friction between the block and floor.



Solution: Given mass of block m= 2 kg, Coefficient of static friction μ_s = 0.4, Force F = 2.5 N

We know the limiting (maximum) force of static friction $f_s = \mu_s R$

$= \mu_s(mg)$ since R= mg

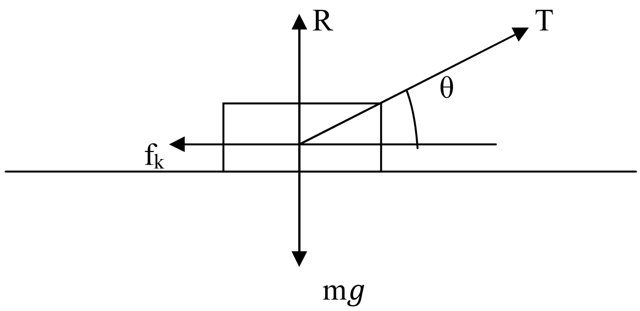
$= 0.4 \times 2 \times 9.8$ (since $g = 9.8$ m/sec^2)

$= 7.84$ N

Obviously, the applied force F is less than the limiting frictional force. Hence, under the force F, the block does not move. Therefore, as long as the block does not move, the (adjustable) frictional force is always equal to the applied force. Thus, the frictional force is 2.5 N.

Example 5: A box of mass m is being pulled across a rough floor by means of a massless rope that makes an angle θ with the horizontal. The coefficient of kinetic friction between the box and the floor is μ_k. What is the tension in the rope when the box moves at a constant velocity?

Solution: Let the tension in the rope be T. All forces acting on the box are shown in the figure.



R is the normal reaction and correspondingly the magnitude of the force of kinetic friction f_k is equal to $\mu_k R$. It is in a direction opposite to the tendency of motion. Since there is no motion in the vertical direction, the resultant of the forces along vertical direction must be zero. Also, as the body moves with a uniform velocity, the resultant force along the horizontal direction is zero. Resolving all the forces along horizontal and vertical direction, we have-

$T \sin\theta + R - mg = 0$ …..(i)

$\mu_k R - T \cos\theta = 0$ …..(ii)

From equation (i), we have-

$R = mg - T \sin\theta$

Putting for R in equation (ii), we get-

$\mu_k (mg - T \sin\theta) - T \cos\theta = 0$

or $\mu_k mg - \mu_k T \sin\theta - T \cos\theta = 0$

or $T = \mu_k mg / (\cos\theta + \mu_k \sin\theta)$

Self Assessment Question (SAQ) 11: Choose the correct option-

(a) The maximum value of static friction is called-

(i) coefficient of static friction (ii) limiting friction (iii) normal reaction (iv) kinetic friction

(b) The limiting friction between two bodies in contact is independent of-

(i) normal reaction (ii) nature of surfaces in contact (iii) the area of surfaces in contact (iv) all of these

(c) The static friction is-

(i) equal to dynamic friction (ii) always less than dynamic friction (iii) always greater than dynamic friction (iv) sometimes greater and sometimes equal to dynamic friction

4.7 LINEAR MOMENTUM

The linear momentum of a particle is defined as the product of the mass of the particle and the velocity of the particle. Usually it is denoted by 'p'.

If m is the mass of a particle and v the velocity then the linear momentum of the particle,

$p = mv$ …..(18)

If kg is the unit of mass and m/sec^2 that of velocity then the unit of linear momentum is given as kg m/sec^2. It is a vector quantity.

In vector form, $\vec{p} = m\vec{v}$ …..(19)

If we have a system of particles of masses $m_1, m_2, m_3, \ldots\ldots m_n$ and velocities $\vec{v_1}, \vec{v_2}, \vec{v_3}, \ldots\ldots \vec{v_n}$, then the total linear momentum of the system would be the vector sum of the momenta of the individual particles i.e.

$\vec{p} = \vec{p_1} + \vec{p_2} + \vec{p_3} + \ldots\ldots\ldots + \vec{p_n}$

$= m_1\vec{v_1} + m_2\vec{v_2} + m_3\vec{v_3} + \ldots\ldots\ldots + m_n\vec{v_n}$

Since $\sum_{i=1}^{n} m_i v_i = M\vec{v_{CM}}$, where M is the total mass and $\vec{v_{CM}}$ is the velocity of the centre of mass of the system.

Thus, $\vec{p} = M\vec{v_{CM}}$ …..(20)

Thus, the total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass. This suggests that the momentum of the system is the same as if all the mass were concentrated at the centre of mass moving with velocity $\overrightarrow{v_{CM}}$. Hence $\vec{v}$ is known as 'system velocity'.

If the system is isolated so that $\vec{p}$ is constant, then

$\overrightarrow{v_{CM}}$ = constant

Thus the centre of mass of an isolated system moves with constant velocity, or is at rest.

4.8 CONSERVATION OF LINEAR MOMENTUM

Let us consider a particle of mass m with initial velocity $\overrightarrow{v_1}$. On applying an external force $\vec{F}$ upon the body, its velocity increases to $\overrightarrow{v_2}$ in a time-interval Δt.

The initial linear momentum of the particle, $\overrightarrow{p_1} = m\overrightarrow{v_1}$

Linear momentum of the particle after the time-interval Δt, $\overrightarrow{p_2} = m\overrightarrow{v_2}$

The change in linear momentum in the time-interval Δt is given as-

$\overrightarrow{p_2} - \overrightarrow{p_1} = m\overrightarrow{v_2} - m\overrightarrow{v_1} = m\,(\overrightarrow{v_2} - \overrightarrow{v_1})$

or $\Delta\vec{p} = m\,\Delta\vec{v}$

Dividing both sides by Δt, we have-

$$\frac{\Delta\vec{p}}{\Delta t} = m\,\frac{\Delta\vec{v}}{\Delta t} \qquad \text{.....(21)}$$

In the above equation, $\frac{\Delta\vec{p}}{\Delta t}$ is the rate of change of linear momentum and $\frac{\Delta\vec{v}}{\Delta t}$ is the rate of change of velocity of the particle which is called the acceleration $\vec{a}$.

Therefore, the above equation can be written as-

$$\frac{\Delta\vec{p}}{\Delta t} = m\vec{a} \qquad \text{.....(22)}$$

But $m\vec{a} = \vec{F}$ (Newton's second law)

Therefore, $$\frac{\Delta\vec{p}}{\Delta t} = \vec{F} \qquad \text{.....(23)}$$

Thus, "the rate of change of linear momentum of a particle is equal to the external force applied on the particle and the change in momentum always takes place in the direction of the force". This is an alternative statement of the Newton's second law.

The equation (23) can be written as-

$$\frac{d\vec{p}}{dt} = \vec{F} \qquad \text{.....(24)}$$

If external force $\vec{F}$ = 0, then

$$\frac{\Delta\vec{p}}{\Delta t} = 0$$

or $\vec{p}$ = constant

i.e. "If the external force acting on a particle is zero, then its linear momentum remains constant". This is known as the 'Principle of Conservation of Linear Momentum'.

For a system of particles, $\overrightarrow{p_1} + \overrightarrow{p_2} + \overrightarrow{p_3} + \ldots\ldots\ldots + \overrightarrow{p_n}$ = constant

4.8.1 Applications of Conservation of Linear Momentum

The conservation of linear momentum governs many phenomena. Few examples are below-

(i) Collisions: When two gross bodies, or two atomic particles collide, the velocities acquired by the bodies after the collision are such that the linear momentum of the system remains conserved. A consideration of conservation of linear momentum shows that in order to slow down the fast neutrons in a reactor, the neutrons should be made to collide with stationary target (nuclei) of nearly the same mass of the neutrons themselves. This is why proton-rich material like paraffin, is a very good moderator.

(ii) Firing of Bullet: Let us consider the firing of bullet from a rifle. Before firing, the linear momentum of the riffle and the bullet is zero. Hence, by conservation of linear momentum, the total momentum of the rifle-bullet system after the firing is also zero. That is, the forward momentum of the bullet is numerically equal to the backward momentum of the rifle.

A rocket works on the principle of conservation of linear momentum.

4.8.2 Newton's Third Law and Conservation of Linear Momentum

When two bodies 1 and 2 (say) collide with each other, then during collision they exert forces on each other. Suppose the exerted force on the body 2 by 1 is $\overrightarrow{F_{12}}$ and that on the body 1 by 2 is $\overrightarrow{F_{21}}$. Suppose, due to these forces, the change in linear momentum of the body 1 is $\Delta\overrightarrow{p_1}$ and that of the body 2 is $\Delta\overrightarrow{p_2}$. Suppose the bodies remain in contact with each other for a time-interval Δt.

During collisions, the bodies are the two parts of one combined body and no external force is acting on this combined body. Hence, by Newton's second law, there should be no change in the momentum of the combined body, i.e.

$$\Delta\overrightarrow{p_1} + \Delta\overrightarrow{p_2} = 0$$

i.e. $\overrightarrow{p_1} + \overrightarrow{p_2}$ = constant

or $m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2}$ = constant

Differentiating with respect to time t, we get-

$$m_1\frac{d\overrightarrow{v_1}}{dt} + m_2\frac{d\overrightarrow{v_2}}{dt} = 0$$

or $m_1\overrightarrow{a_1} + m_2\overrightarrow{a_2} = 0$

where $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$ are the accelerations of body 1 and 2.

or $\overrightarrow{F_{12}} + \overrightarrow{F_{21}} = 0$

or $\overrightarrow{F_{12}} = -\overrightarrow{F_{21}}$

i.e. the force exerted by the body 1 on the body 2 is equal and opposite to the force exerted by the body 2 on the body 1. This is Newton's third law.

4.9 IMPULSE

You can see many occasions in your daily life when a large force is applied on a body for a short time-interval: for example, hitting a cricket-ball by a bat or a ping-pong ball by a stick, striking a nail by a hammer etc. In such cases, the product of the force and the time-interval is called the 'impulse' of the force.

If a constant force $\vec{F}$ is applied on a body for a short interval of time Δt, then the impulse of this force will be $\vec{F}$ x Δt. Impulse is a vector quantity having the direction of force. It may be found by calculating net change in linear momentum.

If $\vec{F}$ is the instantaneous force at time t and it is applied from a time t_1 to a time t_2 , then impulse

$$\vec{I} = \int_{t_1}^{t_2} \vec{F}\,dt \qquad .....(25)$$

Let $\overrightarrow{p_1}$ and $\overrightarrow{p_2}$ be initial and final momenta of body.

By Newton's second law, we know that-

$$\vec{F} = \frac{\overrightarrow{dp}}{dt}$$

Putting for $\vec{F}$ in equation (25), we get-

$$\vec{I} = \int_{t_1}^{t_2} \frac{\overrightarrow{dp}}{dt}\,dt$$

$= \int_{p_1}^{p_2} \overrightarrow{dp} = [\overrightarrow{p}]_{p_1}^{p_2}$

$= \overrightarrow{p_2} - \overrightarrow{p_1}$

= net change in momentum

i.e. the change in momentum is equal to the impulse

or $F \times \Delta t = p_2 - p_1$.

The unit of impulse is Newton-sec.

Example 6: A ball of mass 0.35 kg moving horizontally with a velocity 10 m/sec is struck by a bat. The duration of contact is 10^{-3} sec. After leaving the bat, the speed of the ball is 30 m/sec in a direction opposite to its original direction of motion. Calculate the average force exerted by the bat.

Solution: Given mass of the ball m= 0.35 kg, initial velocity of ball v_1= 10 m/sec, final velocity of ball v_2= 30 m/sec, duration of contact $\Delta t = 10^{-3}$ sec

Change in momentum of ball $\Delta p = p_2 - p_1$

$= mv_2 - mv_1 = m(v_2 - v_1)$

$= 0.35[30-(-10)] = 0.35 \times 40$

= 14 kg m/sec

We know that, Impulse= change in momentum

i.e. $F \times \Delta t = \Delta p$

or $F = \Delta p / \Delta t = 14/10^{-3} = 14000$ Newton

Self Assessment Question (SAQ) 12: A body of mass 2 kg is moving with velocity 10 m/sec. Determine its linear momentum.

Self Assessment Question (SAQ) 13: A body is accelerated from rest by applying a force of 30 N. Calculate the linear momentum of the body after 3 sec.

4.10 SUMMARY

In this unit, you have studied about motion and its causes. To present the clear understanding of motion, some basic definitions like distance, displacement, speed, velocity and acceleration have been discussed. You have studied that the motion of a body is a direct result of its interactions with the other bodies around it. You have also studied Newton's laws of motion i.e. Newton's first law (or law of inertia), Newton's second law (law of change in linear momentum) and Newton's third law (law of action and reaction). According to first law " every body continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it". Newton's second law of motion gives a very important relationship between force and linear momentum and can be expressed as $\vec{F} = \frac{d\vec{p}}{dt}$. For a system of constant mass, it takes the form $\vec{F} = m\vec{a}$. Newton's third law states "to every action there is always an equal and opposite reaction". Forces of action and reaction act on different bodies. In the unit, you have seen that the total linear momentum of a system is conserved if no net external force acts on it which is known as principle or law of conservation of linear momentum. You have also studied the impulse and its relation with linear momentum. Many solved examples are given in the unit to make the concepts clear. To check your progress, self assessment questions (SAQs) are given place to place.

4.11 GLOSSARY

Surroundings – environment, area around a thing or person

Position- location, a place where someone or something is or should be

Specified- particular

Limited- restricted

Confined- restricted

Undergo- suffer

Maintain- sustain

Interactions- exchanges

Resist- refuse to go along with

Friction- resistance

Conservation- protection, preservation or restoration

Assessment- evaluation

4.12 TERMINAL QUESTIONS

1. A ball thrown vertically upward falls at the same place after some time. What is the displacement of the ball?

2. A body is moving on a smooth horizontal surface. Is any force acting on it if it is moving with uniform velocity?

3. The displacement of a particle moving along x-axis is given by $x = a + bt^2$. Find out the acceleration of the particle.

4. A force vector applied on a body is given by $\vec{F} = 2\,\hat{i} - 4\hat{j} + 10\,\hat{k}$ and acquires an acceleration 2 m/sec^2. Find the mass of the body.

5. According to Newton's third law every force is accompanied by an equal and opposite force. How can a movement ever takes place?

6. Calculate the weight of a block of mass 2 kg.

7. A shell is fired from a cannon. The force on the shell is given by $F = 600 - 2 \times 10^5\, t$, where F is in Newton and t in second. The force on the shell becomes zero as soon as it leaves the barrel. Calculate the average impulse imparted to the shell?

8. What is the linear momentum of a 1000 kg car whose velocity is 30 m/sec?

9. The displacement x of a particle moving in one dimension under the action of a constant force is related to time t by equation $t = \sqrt{x} + 5$, where x is in meter and t in second. Find the displacement of the particle when its velocity is zero.

10. An electron starting from rest has a velocity v given by $v = At$, where $A = 3$ m/sec^2 and t is the time. What will be the distance covered by the body in first 2 sec.

11. Mention the difference between distance and displacement.

12. What are Newton's laws of motion? Explain.

13. What is meant by impulse of a force? Prove that the impulse of a force is equal to the change in linear momentum. Give the unit of impulse.

14. What is the principle of conservation of linear momentum? Derive from it Newton's third law of motion.

15. Write notes on-

    (i) Linear momentum (ii) Impulse (iii) Friction

4.13 ANSWERS

Self Assessment Questions (SAQs):

1. Given $x = 2 - 6t + 8t^2$ meter

Differentiating with respect to time t we get-

$v = \frac{dx}{dt} = 0 - 6 + 16\,t$

$= -6 + 16\,t$

For initial velocity of the particle, $t = 0$

Therefore, initial velocity of the particle $v = -6 + 16\,(0)$

$= -6$ m/sec

2. Yes, the body can have zero velocity and finite acceleration. For example, at the highest point of a body thrown vertically upward, the body has zero velocity and acceleration = g

3. Given, displacement of a body $\propto$ time^2

i.e. $x \propto t^2$

or $x = A\,t^2$, where A is a constant

Velocity $v = \frac{dx}{dt} = 2A\,t$

Obviously, the velocity depends on time.

Acceleration $a = \frac{dv}{dt} = 2A$

Obviously, the acceleration of the body is uniform.

4. Given $x = 5t^3 + 6\,t^2 - 5$

Differentiating with respect to time t, we get-

$v = \frac{dx}{dt} = 15\,t^2 + 12\,t$

Acceleration of the particle $a = \frac{dv}{dt} = 30\,t + 12$

Obviously, the acceleration of the particle increases. Therefore, the correct option is (ii).

5. The reaction force acts on the player. Due to large mass (inertia) of the player, the force is not able to make him move.

6. Let F be the force.

$m_1 = 0.5$ kg, $a_1 = 18$ m/sec^2, $a_2 = 6$ m/sec^2

$F = m_1 a_1 = 0.5 \times 18 = 9$ N

Again, $F = m_2 a_2$

or $m_2 = \frac{F}{a_2} = \frac{9}{6} = 1.5$ kg

If both the bodies are fastened together, then total mass $M = m_1 + m_2$

$= 0.5 + 1.5 = 2$ kg

Using $F = M a$

or $a = \frac{F}{M} = \frac{9}{2} = 4.5$ m/sec^2

7. Yes, the ball will return to the hands of the person. The reason is that due to inertia the horizontal velocity of ball remains equal to the velocity of bus.

8. The motion becomes possible since action and reaction, though act simultaneously but on different bodies.

9. (a) (i) , (b) (iv)

10. (i) inertia (ii) force

11. (a) (ii), (b) (iii), (c) (iii)

12. Mass of the body m=2 kg, v = 10 m/sec

Linear momentum $p = mv = 2 \times 10 = 20$ kg m/sec

13. F= 30 N, initial velocity $v_1 = 0$ (since body is at rest initially), $\Delta t = 3$ sec

We know, $F \times \Delta t = p_2 - p_1$

or $F \times \Delta t = p_2 - mv_1$

$30 \times 3 = p_2 - m(0)$

or $90 = p_2$

or $p_2 = 90$

i.e. linear momentum of the body after 3 sec is 90 kg m/sec

Terminal Questions:

1. Zero
2. No
3. Given $x = a + bt^2$

Differentiating with respect to t

$\frac{dx}{dt} = 0 + 2\,bt$

Again differentiating with respect to time

$\frac{d^2x}{dt^2} = 2\,b$

i.e. acceleration a = 2 b

4. Given $\vec{F} = 2\,\hat{i} - 4\hat{j} + 10\,\hat{k}$, acceleration a = 2 m/sec^2

Magnitude of the force F= $|2\,\hat{i} - 4\hat{j} + 10\,\hat{k}| = \sqrt{(2)^2 + (4)^2 + (10)^2}$

$= \sqrt{4 + 16 + 100} = \sqrt{120} = 10.95$ N

Now using F = ma

$m = \frac{F}{a} = \frac{10.95}{2} = 5.48$ kg

5. The motion becomes possible since action and reaction, though act simultaneously but on different bodies.
6. Mass m = 2 kg, g = 9.8 m/sec^2

Weight w = mg

= 2 x 9.8 = 19.6 kg m/sec^2

7. $F = 600 - 2 \times 10^5\, t$

Force F = 0 in time t given by

$0 = 600 - 2 \times 10^5\, t$

or $t = \frac{600}{2 \times 10^5} = 3 \times 10^{-3}$ sec

Impulse of force I = $\int_0^t F\,dt$

$= \int_0^t (600 - 2 \times 10^5\, t)\, dt$

$= 600\text{ t} - \frac{2\times10^5 t^2}{2} = 600\text{ t} - 10^5\text{ t}^2$

$= 600 \text{ x } 3 \text{ x } 10^{-3} - 10^5 \text{ x } (3 \text{ x } 10^{-3})^2$

=0.9 N-sec

8. Mass of car m = 1000 kg, velocity of car v = 30 m/sec

Linear momentum of car p = mv

$= 1000 \text{ x } 30 = 3 \text{ x } 10^4$ kg m/sec

9. Given t = $\sqrt{x} + 3$

or $\sqrt{x}$ = t – 3

Squaring both sides, x = $(t - 3)^2$

or x = $t^2 + 9 - 6t$

or x = $t^2 - 6t + 9$

Differentiating both sides with respect to t, we get-

$\frac{dx}{dt} = 2t - 6$

or v = 2 t – 6

when v = 0, 2 t – 6 =0

or t = 3 sec

At t = 3 sec, displacement x = $(t - 3)^2$

or x = $(3 - 3)^2$

= 0

Hence displacement of particle is zero when its velocity is zero.

10. Given v = At, where A = 3 m/sec^2

v = $\frac{dx}{dt}$ = At

or $dx = At\, dt$

Integrating both sides, we get

$\int_0^x dx = A \int_0^t t\, dt$

or $x = A\, t^2 / 2$

$= 3 \times (2)^2 / 2$

$= 6$ meter

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UNIT 5: PRINCIPLES OF CONSERVATION OF ENERGY AND ANGULAR MOMENTUM.

STRUCTURE:

5.1 INTRODUCTION

In the previous unit we have studied about the motion, causes of motion, Newton's laws of motion and their applications. We have also studied the important law of conservation of linear momentum and its applications. We have also gone into impulse and its relation with change in linear momentum. We often feel that when we execute a motion, some energy is spent and sometimes we say that work is done at the cost of some energy. We know that energy is a very important physical quantity. A dancing, running person is said to be more energetic in comparison of a sleeping, snoring man. In Physics, a moving particle is said to have more energy compared to an identical particle at rest. In the present unit, you will learn about energy and its different kinds in details. We will also study the very important principles of conservation of energy and angular momentum. These principles have very wide applications and are used ordinarily in your Physics courses.

5.2 OBJECTIVES

After studying this unit, you should be able to-

- Compute work done by forces
- apply work-energy theorem
- distinguish between conservative and non-conservative forces
- apply principle of conservation of energy
- solve problems based on conservation of energy
- compute power in various mechanical systems
- solve problems based on conservation of angular momentum

5.3 WORK

The word 'work' has a special meaning in Physics. If a teacher stands near a table and delivers a lecture for some times, then according to the principles of Physics, no work is done. In Physics, work is said to be done when an external force acting on a particle displaces it. The work done by the force on the particle is defined as " the scalar product of the force and the displacement".

If the force $\vec{F}$ displaces a particle by a displacement $\vec{r}$, then the work done by the force $\vec{F}$ is given as-

$$W = \vec{F}.\vec{r} = F\, r \cos\theta \qquad .....(1)$$

where θ is the angle between force $\vec{F}$ and displacement $\vec{r}$.

or work = force x displacement along the direction of force

(r cosθ) is the magnitude of projection of $\vec{r}$ on the force vector $\vec{F}$.

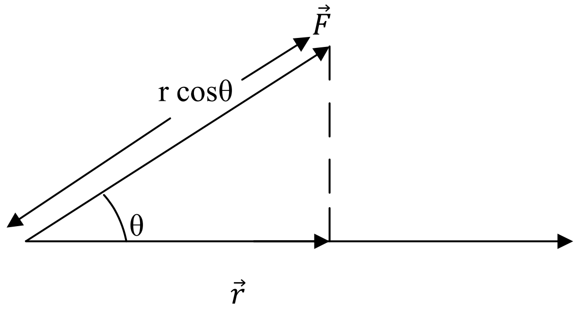


Figure 1

Thus, the work done by a force on a body is defined as the product of magnitudes of force and the component of displacement in the direction of force.

Let us consider a particle P moving along path AB under a force $\vec{F}$ (Figure 2) and this force displaces the particle through an infinitesimal displacement $\overrightarrow{dr}$ then the work done by the force

$$dW = \vec{F}.\overrightarrow{dr} \qquad .....(2)$$

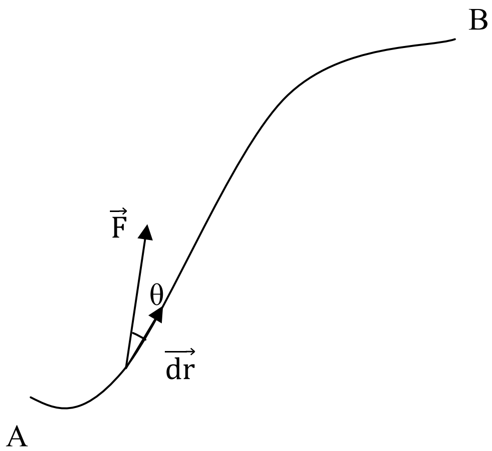


Figure 2

Therefore, total work done by the force in displacing the particle from A to B is given as-

$$W = \int_A^B dW = \int_A^B \vec{F}.\overrightarrow{dr} \qquad .....(3)$$

or $$W = \int_A^B (F\, dr \cos\theta) \qquad .....(4)$$

where θ is the angle between $\vec{F}$ and $\overrightarrow{dr}$. Obviously the work done depends on the magnitudes of force, displacement and the angle between them.

Let us discuss different cases for work done.

If $\theta = 0^0$ i.e. the force vector and displacement vector are parallel to each other, then work done $W = \vec{F}.\vec{r} = F\,r\cos\theta$

$$= F\,r\cos 0^0 = Fr$$

If $\theta = 90^0$ i.e. the displacement of the particle is right angle to the force, then the work done

$W = \vec{F}.\vec{r} = F\,r\cos\theta$

$$= F\,r\cos 90^0 = F\,r\,(0) = 0$$

It means that if a person holding a heavy weight in his hand moves along a level floor, he does no work since the vertical supporting force of his hand is at right angles to the direction of motion. Similarly, you know that when a satellite revolves around the earth, the direction of the force applied by the earth (centripetal force) is always perpendicular to the direction of motion of the satellite. Hence no work is done on the satellite by the centripetal force i.e. centripetal force acting on a body moving in a circle does no work because the force is always at right angles to the direction of motion. Thus the work done in a circular motion is always zero.

You know that if a particle is freely falling vertically then the force of gravity ($m\vec{g}$) acts on the particle vertically downward and the displacement of the particle is also vertically downward. In this case, $\theta = 0^0$, then work done $W = \vec{F}.\vec{r} = F\,r\cos\theta$

$$= F\,r\cos 0^0 = F\,r$$

If force and displacement are opposite in direction i.e. $\theta = 180^0$, then work done

$W = \vec{F}.\vec{r} = F\,r\cos\theta$

$$= F\,r\cos 180^0 = -\,F\,r$$

Thus, you see that if the force is in the same direction as the displacement, the work is positive. If it is opposite to the displacement, the work is negative. Thus, when a person lifts a body from the ground, the work done by the lifting force (upward) of his hand is positive but the work done by the gravitational force (which acts downward) is negative. But on the other hand, when the person lowers the body to ground, the work done by the upward force of his hand is negative but that by the gravitational force is positive. Similarly, when a body slides on a fixed surface, the work done by the frictional force exerted on the body is negative since this force is always opposite to the displacement of the body.

If displacement r = 0, then work done W = F r cosθ = F (0) cosθ = 0 i.e. if the displacement of the particle is zero, then the work done is zero. It means that if the force causes no displacement, the work is zero. A stationary (standing) person holding a heavy weight in his hand may become tired in the physiological sense but according to the principles of Physics, he is not doing any work. Again, the man who has tired to move a luggage but failed, has not done any work because although he has exerted force but the displacement remains zero. As a teacher standing near a table and delivering a lecture, does not do any work according to the principle of Physics.

If the force is varying, then work may be calculated graphically. If we draw a graph between force F and displacement r, then work done by the force F during displacement from r to (r + dr) is given by

$W = \int_{r_1}^{r_2} F\, dr$

= Area enclosed by F- r curve

= Area enclosed PQRS (Figure 3)

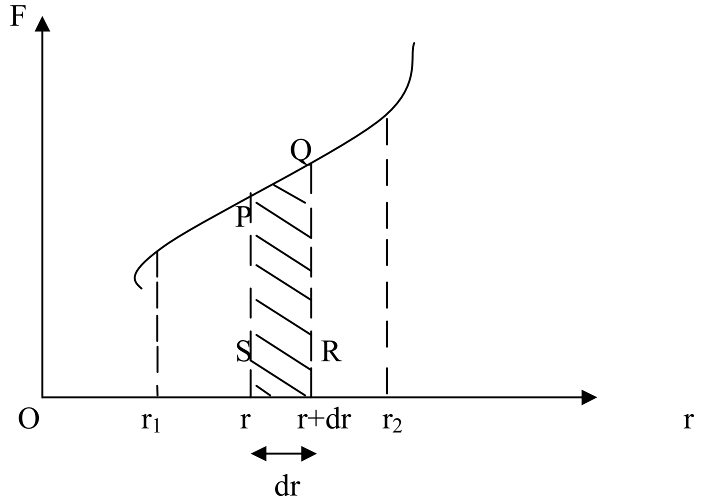


Figure 3

The unit of work is Joule (J). Its other unit is erg.

1 Joule = 10^7 erg

Work is a scalar quantity.

5.3.1 Work in stretching a spring

Let us consider a situation in which one end of a spring is fixed and the other end to a block which can move on a horizontal plane.

Figure 4

Let x = 0 denotes the initial position of the block when the spring is in its original (natural) length (Figure 4 a). Now the block moves from x = 0 to $x = x_1$ by the application of force F (Figure 4 b). We shall calculate the work done on the block in moving from x = 0 to $x = x_1$.

When the spring is stretched slowly, the stretching force increases steadily as the spring elongates i.e. force is variable. When the spring (or block) stretched through a distance $x = x_1$ by applying a force F at the block, the spring on account of its elasticity, exerts a restoring force which acts in the opposite direction of displacement according to Hooke's law of elasticity.

Thus F = - k x …..(5)

where 'k' is called force constant of the spring. The negative sign indicates that the force and the displacement are opposite in direction.

The work done during a small displacement dx can be written as-

$dW = \vec{F}.\overrightarrow{dx}$

$= F\,dx\cos 180^0 = -F\,dx$

$dW = -(-k\,x)\,dx$ (putting for F)

$dW = k\,x\,dx$

The total work done as the block is displaced from x = 0 to $x = x_1$ is

$W = \int_{x=0}^{x=x_1} kxdx = k\left[\frac{x^2}{2}\right]_{x=0}^{x=x_1}$

$= \frac{1}{2} k x_1^2$ .....(6)

If the block moves from $x = x_1$ to $x = x_2$, then total work done will be

$W = \int_{x=x_1}^{x=x_2} kxdx = k\left[\frac{x^2}{2}\right]_{x=x_1}^{x=x_2}$

or $W = \frac{1}{2} k(x_2^2 - x_1^2)$ .....(7)

We should note that if the block is displaced from $x = x_1$ to $x = x_2$ and brought back to $x = x_1$, the work done by the spring force is zero.

Example 1: A block of mass 200 gm is displaced by a distance of 2 m by the application of a force of magnitude 3 N at 30^0. Calculate the work done.

Solution: Given mass of the block m = 200 gm = 0.2 Kg, Displacement r = 2 meter

Force F = 3 N, Angle $\theta = 30^0$

Work done $W = \vec{F}.\vec{r} = F\, r \cos\theta$

$= 3 \times 2 \times \cos 30^0 = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$ Joule = 5.19 Joule

Example 2: Calculate the work done in pulling a spring by 10 cm. The force constant of the spring is 500 N/m.

Solution: Given Increase in length of the spring x_1 = 10 cm = 0.1 meter

Force constant of the spring k = 500 N/m

Work done in pulling the spring $W = \frac{1}{2} k\, x^2$

$= \frac{1}{2} \times 500 \times (0.1)^2 = 250 \times 0.01 = 2.5$ Joule

Example 3: A force F = (10 + 0.2 x) acts on a body in the x-direction, where F is in Newton and x in meter. Find out the work done by this force during a displacement from x = 0 to x = 2 m.

Solution: Force F = (10 + 0.2 x) N, Displacement from x = 0 to x = 2 m, $\theta = 0^0$

Work done $W = \int_{x=0}^{x=2} \vec{F}.\overrightarrow{dx}$

$= \int_{x=0}^{x=2} Fdx \cos\theta = \int_{x=0}^{x=2} Fdx \cos 0^0$

$= \int_{x=0}^{x=2} Fdx = \int_{x=0}^{x=2} (10 + 0.2\, x)\, dx = \left[10\, x + 0.2 \frac{x^2}{2}\right]_0^2$

$= [10\,x + 0.1x^2]_0^2 = \{10 \times 2 + 0.1(2^2)\} - \{10 \times 0 + 0.1(0^2)\}$

= 20.4 Joule

Self Assessment Question (SAQ) 1: A coolie carries a box on his head on a level platform from one place to another. Estimate the work done by the coolie.

Self Assessment Question (SAQ) 2: A car moves with a uniform speed on a smooth level road. Neglecting air resistance, find out the work done by the car.

Self Assessment Question (SAQ) 3: A particle of mass m is moving in a circle of radius r with uniform speed v. What is the work done in a complete revolution? In half revolution?

5.4 POWER

"The time-rate of doing work by an agent or a machine is called power".

Power = $\frac{\text{Work}}{\text{Time}}$

If W is the work done by an agent in t second, then his power P is given as-

$$P = \frac{W}{t} = \frac{\Delta W}{\Delta t} \quad \ldots..(8)$$

Instantaneous power $P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t}$

$$= \frac{dW}{dt} = \frac{d}{dt}(\vec{F}.\vec{r})$$

Since $W = \vec{F}.\vec{r}$

Therefore, $P = \vec{F}.\left(\frac{\overrightarrow{dr}}{dt}\right)$ (if force $\vec{F}$ is constant)

Since $\frac{\overrightarrow{dr}}{dt} = \vec{v}$ (velocity)

Therefore, $P = \vec{F}.\vec{v} \quad \ldots..(9)$

This is the expression for P in the form of scalar product of $\vec{F}$ and $\vec{v}$.

The unit of power is Joule/sec or Watt (W). The other popular units of Power are Horse Power (HP) and Kilo Watt (KW)

1 HP = 746 W and 1KW = 1000 W

If the work done W= 1 Joule and time t = 1 sec

Then power $P = \frac{W}{t} = \frac{1\text{ Joule}}{1\text{ sec}} = 1$ W

i.e. if 1 Joule of work is done in 1 sec, then power is 1 Watt.

The power of a normal person is from 0.05 HP to 0.1 HP.

Example 4: A machine does 50 J work in 2 sec. What is its power?

Solution: Given, W = 50 J, t = 2 sec

Power $P = \frac{W}{t} = \frac{50\text{ Joule}}{2\text{ sec}} = 25$ W

Self Assessment Question (SAQ) 4: A body is moved from rest, along a straight line by a machine delivering constant power. Calculate the distance moved by the body as a function of time t.

5.5 ENERGY

"The capacity of a body to do work is called its energy". The energy is always measured by the work the body is capable of doing. Therefore, the unit of energy is the same as that of work i.e. Joule.

Energy has various forms such as mechanical energy, heat energy, sound energy, chemical energy, light energy, magnetic energy etc. In this unit, we shall concentrate on mechanical energy which includes kinetic energy and potential energy.

5.5.1 Kinetic Energy

"The energy possessed by a body by virtue of its motion is called kinetic energy (K.E.)" i.e. the kinetic energy of a moving body is measured by the amount of work done in bringing the body from the rest position to its present position or which the body can do in going from its present position to the rest position.

Suppose a body of mass m is initially at rest. When we apply a constant force F on the body, it starts moving under an acceleration, then by Newton's second law, we have

F = ma

or $a = \frac{F}{m}$

Suppose, the body acquires a velocity v in time t in moving a distance x.

Using third equation of motion $v^2 = u^2 + 2$ as, we have-

$v^2 = 0 + 2$ ax (here u = 0, s = x)

or $v^2 = 2\frac{F}{m}x$

or $\frac{1}{2}m v^2 = F x$

But Fx is the work done by force F on the body in moving it a distance x. Due to this work the body has itself acquired the capacity of doing work. This is the measure of the kinetic energy of the body. Hence if we represent kinetic energy of a body by K, then

$K = F x = \frac{1}{2}m v^2$

or $K = \frac{1}{2}m v^2$ …..(10)

Kinetic energy $= \frac{1}{2} \times$ mass $\times$ speed^2

Thus the kinetic energy of a moving body is equal to half the product of the mass (m) of the body and the square of its speed (v^2). We see that in this formula v occurs in the second power and so the speed has a larger effect, compared to mass, on the kinetic energy.

We can write equation (10) as-

$K = \frac{1}{2}m\,(\vec{v}.\vec{v})$ …..(11)

The kinetic energy of a system of particles is the sum of the kinetic energies of all its constituent particles i.e.

$K = \sum_i \frac{1}{2} m_i v_i^2$

If a body is initially moving with a uniform speed u and on applying force F on it, its speed increases from u to v in a distance x, then again using third equation of motion $v^2 = u^2 + 2$ as, we have-

$v^2 = u^2 + 2\,ax$

or $v^2 - u^2 = 2\,ax$

$= 2\left(\frac{F}{m}\right)x$ (since $a = \frac{F}{m}$)

or $F x = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

F x is the work done W on the body by the force. Therefore,

$W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$ …..(12)

($\frac{1}{2}$ mv^2 - $\frac{1}{2}$ mu^2) is the increase in the kinetic energy of the body. Thus when a force acts upon a moving body, then the kinetic energy of the body increases and this increase in kinetic energy is equal to the work done.

The unit of kinetic energy is Kg m^2/sec^2 or Joule. It is a scalar quantity like work.

5.5.2 Potential Energy

A body can do work also by virtue of their position or state of strain. **"The energy possessed by a body by virtue of its position or state of strain is called the potential energy (PE) of the body"**. For example, the water at the top of a water-fall can rotate a turbine when falling on it. The water has this capacity by virtue of its position (at a height). Similarly, a wound clock-spring keeps the clock running by virtue of its state of strain. Thus water and wound spring both have potential energy- the former has gravitational potential energy and the later has elastic potential energy.

The potential energy of a body depends on reference level chosen for zero potential energy.

5.5.3 Gravitational Potential Energy

Let us consider that a body of mass m is raised to a height h from earth's surface. In this process work is done against the force of gravity (mg). This work is started in the body in the form of gravitational potential energy U.

This potential energy U = Work done against force of gravity = Weight of the body × height

$$= mg \times h = mgh$$

or $U = mgh$ …..(13)

If this body falls on the earth, an amount mgh of work may be obtained from it.

5.5.4 Elastic Potential Energy

When a spring is stretched or compressed, work has to be done due to the elasticity of the spring. This work is stored as the potential energy of the body which is given by-

$U = \frac{1}{2} kx^2$ …..(14)

where k is force constant of the spring and x is increase in length.

If spring is already stretched by amount x_1 and is further stretched by amount x_2 then work done in stretching the spring from x_1 to x_2 is

$W= \frac{1}{2} k (x_1 + x_2)^2 - \frac{1}{2} kx_1^2$

$= \frac{1}{2} k (x_1^2 + x_2^2 + 2x_1x_2 – x_1^2)$

$= \frac{1}{2} kx_2 (x_2 + 2x_1)$

Therefore, increase in potential energy $\Delta U = W = \frac{1}{2} kx_2 (x_2 + 2x_1)$ …..(15)

Example 5: A spring obeys Hooke's law with a force constant 800 N/m. If it is stretched through 10 cm, how much work is required in this process?

Solution: Given, Force constant k = 800 N/m, x = 10 cm = 0.1 m

Required work $W = \frac{1}{2} kx^2$

$= \frac{1}{2} \times 800 \times (0.1)^2 = 4$ Joule

Example 6: A man ascends to a temple from ground level a vertical rise of 1,800 meter. His mass is 50 Kg. He takes 6 hours. What is the average power exerted?

Solution: Given, h= 1,800 meter, m= 50 Kg , t = 6 hours = 6 × 60 × 60 = 21,600 sec

$P = \frac{W}{t} = \frac{mgh}{t} = \frac{50 \times 9.8 \times 1800}{21600} = 40.83$ W

Example 7: A car of mass 200 Kg is running with a uniform speed of 80 Km/hour. Calculate its kinetic energy.

Solution: Given, mass of car m = 200 Kg, v = 80 Km/Hour $= \frac{80 \times 1000}{60 \times 60} = 22.22$ m/sec

Kinetic energy $K = \frac{1}{2} m v^2$

$= \frac{1}{2} \times 200 \times (22.22)^2 = 49372.84$ Joule

Self Assessment Question (SAQ) 5: What happens to the mechanical energy which is spent in raising a heavy body from a lower to a higher level?

Self Assessment Question (SAQ) 6: Does the work done in raising a box onto a platform depend on how fast is it raised?

Self Assessment Question (SAQ) 7: Establish the relation between kinetic energy and linear momentum.

Self Assessment Question (SAQ) 8: Choose the correct option-

(i) Of the following the one possessing the kinetic energy is-

(a) water stored in a dam (b) a bullet in flight (c) stretched rubber band

(d) air in bicycle pump

(ii) The kinetic energy of a moving body varies with mass directly-

(a) m^{-1} (b) m^2 (c) m (d) m^0

(iii) The kinetic energy of a body of mass 1 Kg and momentum 4 N-sec is-

(a) 8 J (b) 7 J (c) 16 J (d) 0 J

5.6 WORK-ENERGY THEOREM

Work-Energy theorem states that **"the work done by the net force acting on a body is equal to the change in kinetic energy of a body".** i.e.

Work = gain in kinetic energy

$$= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

Let us consider a body of mass m acted upon a net force F along x-axis. If body moves from a position x_1 to position x_2 along the x-axis, its velocity increases from v_1 and v_2. The work done by the force in this displacement is –

$$W = \int_{x_1}^{x_2} F\, dx \qquad \text{.....(16)}$$

By Newton's second law, we know

$$F = ma = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx} \qquad \left(\text{Putting } \frac{dx}{dt} = v\right)$$

Therefore, $W = \int_{x_1}^{x_2} mv\frac{dv}{dx}\, dx$

$$= m\int_{x_1}^{x_2} v\frac{dv}{dx}\, dx = m\int_{v_1}^{v_2} v\,dv$$

$$= m\left[\frac{v^2}{2}\right]_{v_1}^{v_2} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

or $W = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$ .....(17)

or $W = K_2 - K_1$ .....(18)

where K_1 and K_2 are the initial and final kinetic energies of the body. Thus, if ΔK represents the change in kinetic energy, $\Delta K = K_2 - K_1$ then, we have-

$$W = \Delta K \quad \text{.....(19)}$$

This is the mathematical statement of work-energy theorem.

5.7 CONSERVATIVE AND NON-CONSERVATIVE FORCES

We divide the forces in two categories- conservative forces and non-conservative forces. We can define these forces in various ways.

"A force acting on a particle is conservative if the particles after going through a complete round trip, returns to its initial position with the same kinetic energy as it had initially". Let us understand this with some examples.

When we throw a ball upward against the gravity of earth, the ball reaches a certain height coming momentarily to rest so that its kinetic energy becomes zero, then it returns to our hand under gravity with same kinetic energy with which it was thrown (assuming air resistance to be zero). Thus the force of gravity is conservative force.

Similarly, elastic force exerted by an ideal spring is conservative. The electrostatic force is also conservative.

"A force acting on a particle is non-conservative if the particle, after going through a complete round trip, returns to its initial position with changed kinetic energy".

In the above example, we have assumed the air-resistance to be zero. Actually, air-resistance (viscous force) is always there. This force always opposes the motion. Therefore, a part of kinetic energy is always spent in overcoming this force. Hence, the ball returns to hand with smaller kinetic energy than it had initially. Obviously, this viscous force which is responsible for the decrease in kinetic energy is non-conservative force. Similarly, frictional force between two bodies or planes is a non-conservative force. The force of induction is also an example of non-conservative force.

We can also distinguish between conservative and non-conservative forces in terms of work done. **"A force acting on a particle is conservative if the net work done by the force in a complete round trip of the particle is zero, if the net work done is not zero, then the force is non-conservative".**

Yet, there is a third way of distinguishing between conservative and non-conservative forces. Suppose a particle acted upon by a conservative force goes from P to Q along path 1 and returns to P along path 2 (Figure 5 a). As the force is conservative, the work done in the outgoing journey is equal and opposite to that in the return journey the work done in complete round trip which is equal to the net gain in kinetic energy is zero as defined earlier. Hence i.e. $W_{P\rightarrow 1\rightarrow Q} = -W_{Q\rightarrow 2\rightarrow P}$ …..(20)

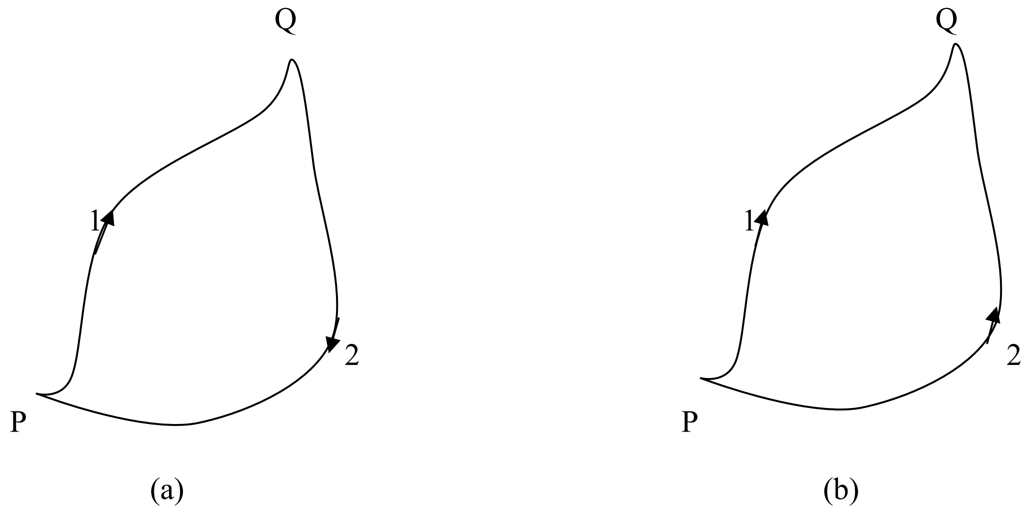


Figure 5

If the particle be moved from P to Q along the path 2 (Figure 5 b), the work done would be equal and opposite to that in moving from Q to P along the path 2 i.e.

$W_{P\rightarrow 2\rightarrow Q} = -W_{Q\rightarrow 2\rightarrow P}$ …..(21)

Comparing equations (20) and (21), we get-

$W_{P\rightarrow 1\rightarrow Q} = W_{P\rightarrow 2\rightarrow Q}$ …..(22)

This shows that the work done by the conservative force in moving the particle from P to Q along the path 1 is the same as that along the path 2. Thus if the work done by a force depends only on the initial and final states and not on the path taken, it is called a conservative force.

5.8 CONSERVATION OF ENERGY

"If a system is acted on by conservative forces, the total mechanical energy of the system remains constant". This is called the principle of conservation of energy.

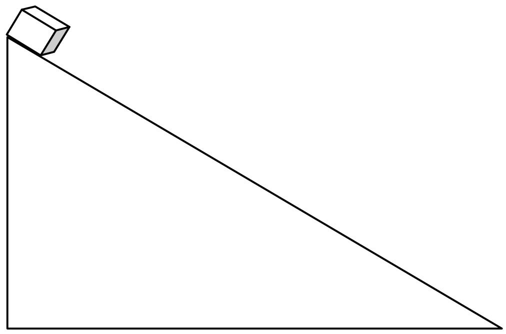
i.e. Mechanical Energy E = Kinetic Energy (K.E.) + Potential Energy (P.E.)

= Constant

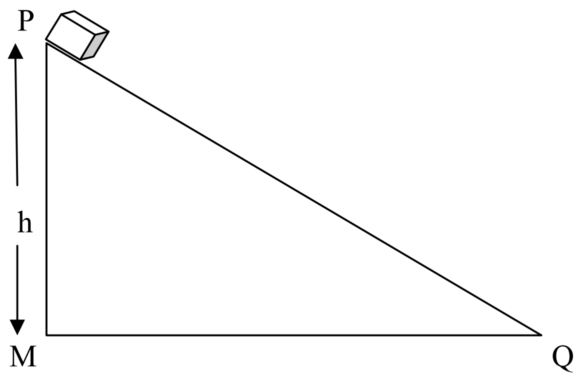
or $E = K + U =$ Constant (under conservative force) …..(23)

The total mechanical energy (K + U) is not constant if non-conservative forces such as friction, act between the parts of the system. We cannot apply the principle of conservation of energy in presence of non-conservative forces and a more general law stated as "The total energy of the universe remains constant" holds. This simply means that the energy may be transformed from one form to another. For example, in loudspeakers, electric bells; the electrical energy is converted into sound energy while in electromagnet, electrical energy is converted into magnetic energy and for a ball falling on earth, the mechanical energy is converted into heat energy etc. The work-energy theorem is still valid even in the presence of non-conservative forces.

Example 8: A block of mass m slides along a frictionless surface as shown in figure. If it is released from rest at P, what is its speed at Q?



Solution:



Let v be the speed of block at Q.

Applying principle of conservation of energy, Total mechanical energy at P = Total mechanical energy at Q

Kinetic energy at P + Potential energy at P = Kinetic energy at Q + Potential energy at Q

Considering MQ as reference level –

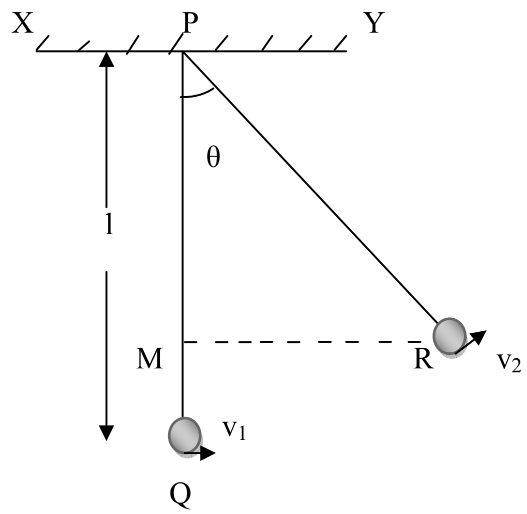
$\frac{1}{2}$ m $(0)^2$ + mgh = $\frac{1}{2}$ m v^2 + 0

mgh = $\frac{1}{2}$ m v^2

$v = \sqrt{2gh}$

Example 9: A pendulum bob has a speed 3 m/sec while passing through its lowest position. What is its speed when it makes an angle of 60^0 with the vertical? The length of the pendulum is 0.5 m. Take g = 10 m/sec^2

Solution:



Let v_2 be the speed of the bob when it makes an angle of 60^0 with the vertical.

Applying principle of conservation of energy-

Total mechanical energy at Q = Total mechanical energy at R

Kinetic energy at Q + Potential energy at Q = Kinetic energy at R + Potential energy at R

Considering XY as reference level –

$\frac{1}{2} mv_1^2 + mgl = \frac{1}{2} mv_2^2 + mg\,(PM)$ …..(i)

In right angle triangle PMR-

PM = PR cosθ

$= l \cos\theta$

Putting for PM in equation (i), we get-

$\frac{1}{2} mv_1^2 + mgl = \frac{1}{2} mv_2^2 + mg \ (l \cos\theta)$

$\frac{1}{2} v_1^2 + gl = \frac{1}{2} v_2^2 + g \ (l \cos\theta)$

or $v_1^2 + 2 gl = v_2^2 + 2g l \cos\theta$

or $v_2^2 = v_1^2 + 2 gl - 2g l \cos\theta$

or $v_2 = \sqrt{v_1^2 + 2 gl - 2g l \cos\theta}$

Here $v_1 = 3$ m/sec, $l = 0.5$ m, $g = 10$ m/sec^2, $\theta = 60^0$

Therefore, $v^2 = \sqrt{3^2 + 2 \times 10 \times 0.5 - 2 \times 10 \times 0.5\cos 60^0}$

$= \sqrt{9 + 10 - 10 \times \frac{1}{2}} \quad = \sqrt{19 - 5}$

$= \sqrt{14} = 3.74$ m/sec

Self Assessment Question (SAQ) 9: If energy is neither created nor destroyed, what happens to the so much energy spent against friction?

Self Assessment Question (SAQ) 10: What happens to the mechanical energy which is spent in raising a heavy body from a lower to a higher level?

5.9 ANGULAR MOMENTUM

"The moment of linear momentum of a particle rotating about an axis is called angular momentum of the particle". It is denoted by J.

If a particle be rotating about an axis of rotation, then

J = linear momentum × distance

$= p \times r$

$= mv \times r$ (since $p = mv$)

or $J = mvr$

where m, v and r are the mass of the particle, linear velocity and distance of particle from axis of rotation respectively.

But $v = r\omega$, where ω is the angular velocity

Therefore, $J = m\,(r\omega)\,r = mr^2\,\omega$

$$\text{or} \quad J = I\,\omega \qquad \text{.....(24)}$$

where $mr^2 = I$ = Moment of Inertia of particle about the axis of rotation

Let us suppose a body be rotating about an axis with an angular velocity ω. All the particles of the body will have the same angular velocity ω but different linear velocities.

Let a particle be at a distance r_1 from the axis of rotation, the linear velocity of this particle is given by-

$v_1 = r_1\omega$

If m_1 be the mass of the particle, then its linear momentum $p_1 = m_1v_1$

The moment of this momentum about the axis of rotation i.e. angular momentum of the particle J_1 = linear momentum × distance

$= p_1 \times r_1$

$= m_1v_1 \times r_1$

$= m_1(r_1\omega) \times r_1$

$= m_1r_1^2\,\omega$

Similarly, if the masses of other particles be m_2, m_3, …… and their respective distances from the axis of rotation be r_2, r_3, ……… , then the moments of their linear momenta about the axis of rotation will be $m_2r_2^2\,\omega$, $m_3r_3^2\,\omega$, ………..respectively. The sum of these moments of linear momenta of all the particles i.e. the angular momentum of the body is given by-

$J = m_1r_1^2\,\omega + m_2r_2^2\,\omega + m_3r_3^2\,\omega + \ldots\ldots\ldots$

$= (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots\ldots)\omega$

$= (\sum mr^2)\omega$

$$\text{or} \quad J = (\sum mr^2)\omega \qquad \text{.....(25)}$$

But $(\sum mr^2) = I$, is the moment of inertia of the body about the axis of rotation and plays the same role in rational motion as mass plays in linear motion. This will be discussed in unit 6.

Therefore, angular momentum J = I ω …..(26)

The unit of angular momentum is Kg-m^2/sec. It is a vector quantity.

In vector form, $\vec{J} = \vec{r} \times \vec{p}$ …..(27)

or $\vec{J}$ = r p sinθ n^ …..(28)

where θ is the angle between $\vec{r}$ and $\vec{p}$ and n^ is the unit vector perpendicular to the plane containing $\vec{r}$ and $\vec{p}$.

Magnitude of angular momentum J = r p sinθ …..(29)

5.10 CONSERVATION OF ANGULAR MOMENTUM

We know that relation for angular momentum-

J = I ω …..(30)

The rate of change of angular momentum-

$$\frac{\Delta J}{\Delta t} = I\,\frac{\Delta \omega}{\Delta t}$$

$$= I\,\alpha \qquad \text{…..(31)}$$

(since $\frac{\Delta\omega}{\Delta t}$ = α, angular acceleration)

But I α = τ (Torque), in analogy with Newton's law for linear motion Ma= F(Force).

Therefore, $\frac{\Delta J}{\Delta t} = \tau$ …..(32)

i.e. the time-rate of change of angular momentum of a body is equal to the external torque acting upon the body. Actually, a torque is required for rational motion just as a force is needed to cause a linear motion.

If τ = 0, then $\frac{\Delta J}{\Delta t}$ = 0

or ΔJ = 0

or J = Constant

or I ω = Constant …..(33)

i.e. **"If no external torque is acting upon a body rotating about an axis, then the angular momentum of the body remains constant"**. This is called the law of conservation of angular momentum. If I decreases, ω increases and vice-versa.

In vector form-

$\vec{J} = \vec{r} \times \vec{p}$

Differentiating both sides with respect to time t, we get-

$\frac{d\vec{J}}{dt} = \frac{d}{dt}\,(\vec{r} \times \vec{p})$

$= (\frac{d\vec{r}}{dt} \times \vec{p}) + (\vec{r} \times \frac{d\vec{p}}{dt})$

$= (\vec{v} \times m\vec{v}) + (\vec{r} \times \frac{d\vec{p}}{dt})$ (since $\frac{d\vec{r}}{dt} = \vec{v}$ and $\vec{p} = m\vec{v}$)

$= m(\vec{v} \times \vec{v}) + (\vec{r} \times \frac{d\vec{p}}{dt})$

$= 0 + (\vec{r} \times \frac{d\vec{p}}{dt})$ (since $\vec{v} \times \vec{v} = 0$)

or $\frac{d\vec{J}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$ …..(34)

By Newton's second law, $\frac{d\vec{p}}{dt} = \vec{F}$

Therefore, $\frac{d\vec{J}}{dt} = \vec{r} \times \vec{F}$ …..(35)

But $\vec{r} \times \vec{F} = \vec{\tau}$, the torque acting on the particle

Therefore, equation (35) becomes-

$\frac{d\vec{J}}{dt} = \vec{\tau}$ …..(36)

i.e. the time-rate of change of angular momentum of a particle is equal to the torque acting on the particle.

If $\vec{\tau} = 0$, then $\frac{d\vec{J}}{dt} = 0$

i.e. $\vec{J}$ = Constant …..(37)

Let us discuss some examples based on conservation of angular momentum.

When a diver jumps into water from a height, he does not keep his body straight but pulls in his arms and legs towards the centre of his body. On doing so the moment of inertia I of his body decreases. Since the angular momentum I ω remains constant, therefore, on decreasing I, his angular velocity ω correspondingly increases. Hence jumping he can rotate his body in the air.

Suppose a man with his arms outstretched and holding heavy dumb-bells in each hand, is standing at the centre of a rotating table. When the man pulls in his arms, the speed of rotation of the table increases. This is why? The reason is that on pulling in the arms, the distance of the dumbbells from the axis of rotation decreases and therefore, the moment of inertia I of the man decreases. But according to conservation of angular momentum, the total angular momentum remains constant. Therefore, on decreasing moment of inertia I, the angular velocity ω increases.

Example 10: A mass of 3 Kg is rotating on a circular path of radius 1.0 m with angular velocity of 40 radian/sec. If the radius of the path becomes 0.8 m, what will be the value of angular velocity?

Solution: Given, m = 3 Kg, r_1 = 1.0 m, ω_1 = 40 radian/sec, r_2= 0.8 m

Using law of conservation of angular momentum-

$I_1 \omega_1 = I_2 \omega_2$

or $(mr_1^2)\, \omega_1 = (mr_2^2)\, \omega_2$

or $r_1^2\, \omega_1 = r_2^2\, \omega_2$

$(1)^2 (40) = (0.8)^2\, \omega_2$

or ω_2 = 62.5 radian/sec

Self Assessment Question (SAQ) 11: If the earth suddenly contracts to half its radius, what would be the length of the day? By how much would the duration of day be decreased?

Self Assessment Question (SAQ) 12: Choose the correct option-

(i) Angular momentum of a body is the product of-

(a) linear velocity and angular velocity (b) mass and angular velocity

(c) centripetal force and radius (d) moment of inertia and angular velocity

(ii) When a torque acting on a system is zero, what is conserved-

(a) angular velocity (b) linear momentum (c) force (d) angular momentum

5.11 SUMMARY

In the present unit, we have studied about work, power, energy, work-energy theorem, conservative and non-conservative forces, conservation of energy and angular momentum. We have proved that work done on a particle is equal to the change in its kinetic energy. We have also studied the differences between conservative and non-conservative forces. A force is called conservative if the work done by it during a round trip of a system is always zero otherwise non-conservative. The force of gravitation, electrostatic force, force by a spring etc. are conservative forces while friction is an example of non-conservative force. In this unit, we have concentrated on mechanical energy. In the unit, we have studied the principle of conservation of energy according to which, the total mechanical energy of a system is conserved if the system is acted on by conservative forces. We have also focused on angular momentum and its conservation with few examples. You have seen that if there is no external torque acted on a system then the angular momentum of the system is conserved. To make the concepts more clear, many solved examples are incorporated in the unit. To check your progress, self assessment questions (SAQs) are given in the unit.

5.12 GLOSSARY

Execute – perform a skilful action

Energy – the strength and vitality required to keep active

Conservation- preservation or restoration of the natural environment

Distinguish – recognize or treat as different

External- relating to the outside of something

Displace- move from the proper or usual position

Centripetal – moving towards a centre

Exert- apply or bring to bear a force

Stretch- be able to be made longer or wider without tearing or breaking

Mechanical- relating to or operated by a machine or machinery

5.13 TERMINAL QUESTIONS

1. Is it possible that a body be in accelerated motion under a force acting on the body yet no work is being done by the force? Give example.

2. A particle moves along x-axis from x = 0 to x = 4 m under the influence of a force $F = (5 - 2x + 3x^2)$ N. Calculate the work done in this process.

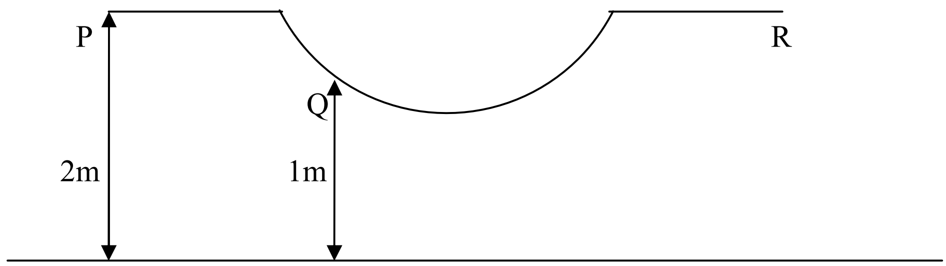
3. A block is pushed through 4 m across a floor offering 50 N resistance. How much work is done by the resisting force?

4. Calculate the work done by a force $F = kx^2$ acting on a particle at an angle of 60^0 with x-axis to displace it from 1 m to 3 m along the x-axis.

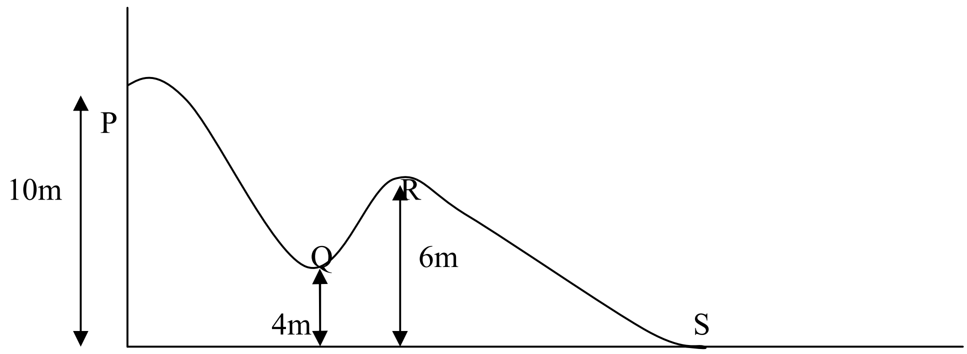
5. If the mass of a body is reduced to half and its velocity is doubled, then what will be ratio of kinetic energy?

6. A boy whose mass is 51 Kg climbs with constant speed, a vertical rope 6 m long in 10 sec. How much work does the boy perform? What is his power output during the climb?

7. A particle is placed at the point P of a frictionless track PQR as shown in figure. It is pushed slightly towards right. Find its speed when it reaches the point Q. Take $g = 10\ m/sec^2$



8. The given figure shows the vertical section of a frictionless surface. A block of mass 2 Kg is released from position P. Compute its kinetic energy as it reaches positions Q, R and S.



9. A ball tied to a string takes 2 sec in one complete revolution in a horizontal circle. If by pulling the cord, the radius of the circle is reduced to half of the previous value, then how much time the ball will now take in one revolution?

10. What is the meaning of work in Physics? What should be the angle between the force and the displacement for maximum and minimum work?

11. Prove and discuss work-energy theorem.

12. Define and explain the difference between conservative and non-conservative force.

13. What is energy? Discuss the different types of mechanical energy with examples.

14. What is the principle of conservation of energy?

15. Define angular momentum. How is the angular momentum of a body conserved? Explain.

5.14 ANSWERS

Self Assessment Questions (SAQs):

1. According to principle of Physics, work = force × displacement along the direction of force. The weight of the box acts vertically downward and the displacement is horizontal i.e. the angle between force of gravity and displacement is 90^0, therefore, $W = Fr\cos\theta = Fr\cos 90^0 = Fr(0) = 0$, i.e. the coolie does no work.

2. The weight of the car (i.e. force of gravity) and displacement of car are at right angle i.e. $\theta = 90^0$, therefore work done $W = Fr\cos\theta = Fr\cos 90^0 = 0$, i.e. work done by car is zero.

3. We know that in circular motion the displacement and force (centripetal force) are at right angle always, therefore the work done by the centripetal force is always zero.

4. Given, P = Constant i.e. P = Fv Constant

But F = ma, therefore, P = (ma)v

or P = mav …..(i)

Using first equation of motion v = u + at

v = 0 + at = at (since u = 0)

v = at

Therefore, $P = (ma)at = ma^2t$

or $a^2t = \frac{P}{m}$ = Constant

or a^2t = Constant .....(ii)

Using second equation of motion $s = ut + \frac{1}{2}at^2$

$s = 0\times t + \frac{1}{2}at^2$

$s = \frac{1}{2}at^2$

Squaring both sides, $s^2 = \frac{1}{4}a^2t^4$

or $s^2 = \frac{1}{4}(a^2t)t^3$

or $s^2 \propto t^3$ (since a^2t is constant)

5. The mechanical energy spent in raising a heavy body from a lower to a higher level is not lost but is stored in the form of potential energy.

6. No, since work done W = mgh

7. We know, $K = \frac{1}{2}mv^2$

Multiplying by m in numerator and denominator of RHS, we get-

$K = \frac{1}{2}\frac{m^2v^2}{m}$

or $K = \frac{1}{2}\frac{(mv)^2}{m} = \frac{p^2}{2m}$

8. (i) (b), (ii) (c), (iii) (a)

9. The energy is dissipated in the form of heat. The heat energy so produced is not available for work.

10. By the law of conservation of angular momentum-

$I\,\omega$ = Constant

or $I_1\omega_1 = I_2\omega_2$

Here, $I_1 = \frac{2}{5}MR_1^2$, $\omega_1 = \frac{2\pi}{T_1} = \frac{2\pi}{24} = \frac{\pi}{12}$ radian/hour, $I_2 = \frac{2}{5}MR_2^2 = \frac{2}{5}M\left(\frac{R_1}{2}\right)^2 = \frac{MR_1^2}{10}$, $\omega_2 = \frac{2\pi}{T_2}$

Therefore, $\left(\frac{2}{5}MR_1^2\right)\left(\frac{\pi}{12}\right) = \left(\frac{MR_1^2}{10}\right)\left(\frac{2\pi}{T_2}\right)$

or T_2 = 6 hours

Decrease in duration of day = 24 – 6 = 18 hours

11. (i) (d), (ii) (d)

Terminal Questions:

1. Yes, it is possible in a uniform circular motion of a body since instantaneous velocity is always perpendicular to force. For example, moon revolves about the earth under centripetal force and gets accelerated but work done is zero.

2. Given, Force $F = (5 - 2x + 3x^2)$ N

Work done $W = \int_{x=0}^{x=4} \vec{F}.\overrightarrow{dx}$

$= \int_{x=0}^{x=4} Fdx\cos\theta = \int_{x=0}^{x=4} Fdx\cos 0^0$ (since $\theta = 0^0$)

$= \int_0^4 Fdx = \int_0^4 (5 - 2x + 3x^2)\, dx = \left[5x - \frac{2x^2}{2} + \frac{3x^3}{3}\right]_0^4$

= 68 N

3. Given, r = 4 m, F = 50 N, Obviously here $\theta = 0^0$

Work done $W = \vec{F}.\vec{r} = Fr\cos\theta$

$= 50 \times 4\cos 0^0 = 200$ Joules

4. Given $F = kx^2$, $\theta = 60^0$, $x_1 = 1$ m, $x_2 = 3$ m

We know $W = \int_{r_1}^{r_2} \vec{F}.\overrightarrow{dx} = \int_{r_1}^{r_2} Fdx\cos\theta$

$= \int_{x_1=1}^{x_2=3} kx^2 \cos 60^0 dx = \int_1^3 (kx^2)\frac{1}{2}dx$

$= \frac{k}{2}\int_1^3 x^2 dx = \frac{k}{2}\left[\frac{x^3}{3}\right]_1^3 = 4.33$ k

5. Let $m_1 = m$, $v_1 = v$

Hence $m_2 = m/2$, $v_2 = 2v$

$K_1 = ½(m_1v_1^2)$, $K_2 = ½(m_2v_2^2)$

Ratio of kinetic energies $K_1/K_2 = ½(m_1v_1^2)/ ½(m_2v_2^2)$

$= m_1v_1^2/ m_2v_2^2 = mv^2/\{(m/2)(2v)^2\} = ½$

i.e. kinetic energy becomes doubled.

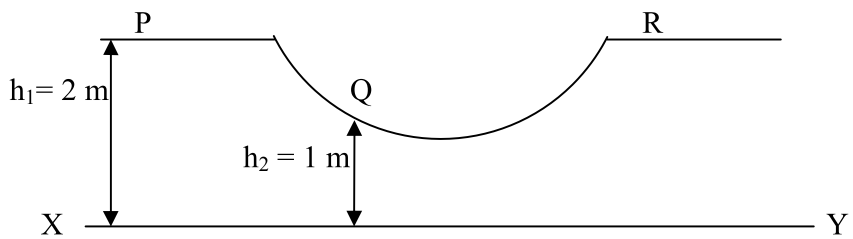
6. Given, m = 51 Kg, t= 10 sec, h= 6 m

The boy does work against his weight (i.e. gravitational force) in climbing. Therefore,

Work done W = mgh = $51 \times 9.8 \times 6$ = 2998.8 Joule

This work is done in 10 sec. Therefore the power output P = $\frac{W}{t} = \frac{2998.8}{10}$ = 299.88 W

7.



Let the speed of particle at Q be v. Obviously, the kinetic energy of particle at P will be zero.

Applying principle of conservation of energy-

Total mechanical energy at P = Total mechanical energy at Q

Kinetic energy at P + Potential energy at P = Kinetic energy at Q + Potential energy at Q

Considering horizontal surface XY as reference level –

$0 + mgh_1 = \frac{1}{2} mv^2 + mgh_2$

or $gh_1 = \frac{1}{2} v^2 + gh_2$

or $v^2 = 2gh_1 - 2gh_2$

or $v = \sqrt{2gh_1 - 2gh_2} = \sqrt{2g(h_1 - h_2)} = \sqrt{2 \times 10(2-1)}$ = 4.47 m/sec

8.

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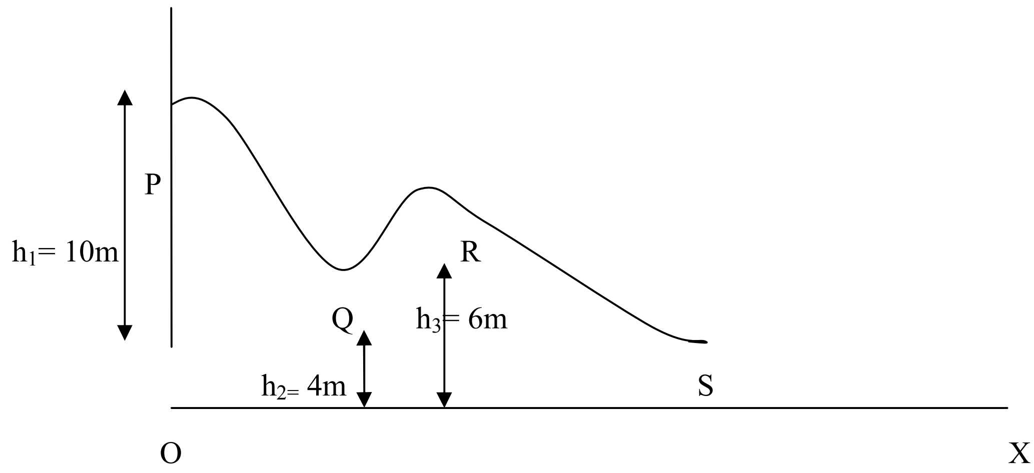

Applying principle of conservation of energy-

Total mechanical energy at P = Total mechanical energy at Q

Kinetic energy at P + Potential energy at P = Kinetic energy at Q + Potential energy at Q

Considering horizontal plane XY as reference level –

$0 + mgh_1 = K_Q + mgh_2$

$K_Q = mgh_1 - mgh_2 = mg(h_1 - h_2)$

$= 2\times 9.8(10-4) = 117.6$ Joule

Again using principle of conservation of energy-

Total mechanical energy at P = Total mechanical energy at R

Kinetic energy at P + Potential energy at P = Kinetic energy at R + Potential energy at R

Considering horizontal plane XY as reference level –

$0 + mgh_1 = K_R + mgh_3$

$K_R = mg(h_1 - h_3) = 2 \times 9.8(10-6) = 78.4$ Joule

Applying principle of conservation of energy-

Total mechanical energy at P = Total mechanical energy at S

Kinetic energy at P + Potential energy at P = Kinetic energy at S + Potential energy at S

Considering horizontal plane XY as reference level –

$0 + mgh_1 = K_S + 0$

$K_S = mgh_1 = 2 \times 9.8 \times 10 = 196$ Joule

9. By the conservation of angular momentum-

$I_1\omega_1 = I_2\,\omega_2$

Here $I_1 = mr_1^2$, $\omega_1 = \frac{2\pi}{T_1} = \frac{2\pi}{2} = \pi$ radian/sec, $I_2 = m\,(\frac{r_1}{2})^2 = \frac{mr_1^2}{4}$

or $mr_1^2 \times \pi = \frac{mr_1^2}{4} \times \omega_2$

or $\omega_2 = 4\pi$

or $\frac{2\pi}{T_2} = 4\pi$

or $T_2 = 0.5$ sec

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5.16 SUGGESTED READINGS

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UNIT 6: ROTATIONAL MOTION

STRUCTURE:

6.1 INTRODUCTION

In previous units 4 and 5, you have studied some important concepts of Mechanics such as displacement, velocity, acceleration, causes of motion, Newton's laws of motion, linear momentum, work, power, energy etc. You have also studied important conservation principles such as conservation of linear momentum, conservation of energy and conservation of angular momentum. In these units, you have dealt with mainly translatory motion. We have not gone into describing and analyzing the rotational motion of the particles. You should know that rotational (angular) motion also plays an important role in this universe. You have many examples of rotational motion in your life. Rotating galaxies, orbiting planets, bicycle wheels, train wheels, pulleys, door of almirah, ceiling fan in your room etc. have rotational motion involved. Obviously, it is very necessary to study and analyze the rotational motion of particles. Therefore, in this unit we shall study angular velocity, angular acceleration, equations of angular motion, angular momentum and torque. In this unit, we shall also study some important examples and applications based on rotational motion.

6.2 OBJECTIVES

After studying this unit, you should be able to-

- Compute angular velocity and angular acceleration of a particle undergoing rotational motion
- apply equations of angular motion
- relate torque and moment of inertia
- relate linear and angular variables of rotating body
- compute rotational kinetic energy and angular momentum of particles
- solve problems based on rotational motion
- compare linear quantities and angular quantities

6.3 ROTATIONAL MOTION

If an external force applied to a body does not produce any displacement of the particles of the body relative to each other, then the body is called a 'rigid body'. In fact, no real body is perfectly rigid. However, in solid bodies, except rubber etc., the relative displacement by the external force is so small that it can be neglected. Hence usually when we speak of a body, we mean a rigid body.

When a body rotates about a fixed axis, the rotation is known as 'rotatory motion' or 'angular motion' and the axis is known as the 'axis of rotation'. In rotatory motion, every particle of the body moves in a circle and the centres of all these circles lie at the axis of rotation. The rotating blades of an electric fan and the motion of a top are the examples of rotatory motion.

Consider the door of your almirah. When you open the door, the vertical line passing through the hinges is held fixed and that is the axis of rotation. Each particle of the door describes a circle with the centre at the foot of the perpendicular from the particle on the axis. All these circles are horizontal and thus perpendicular to the axis.

Look at the ceiling fan in your room. When it is on, each point on its body goes in a circle. Locate the centres of the circles traced by different particles on the three blades of the fan and the body covering the motor. All these centres lie on a vertical line through the centre of the body. The fan rotates about this vertical line.

6.3.1 Angular Displacement

Let us consider that a particle P is moving in a circle around a point O with a constant speed v. Let X is the initial position of the particle at time t = 0. The instantaneous position of P is expressed by an angle θ between a radial line OP and a reference line OX. P is the position of the particle after time t. θ is the angle subtended at the centre O by particle in time t.

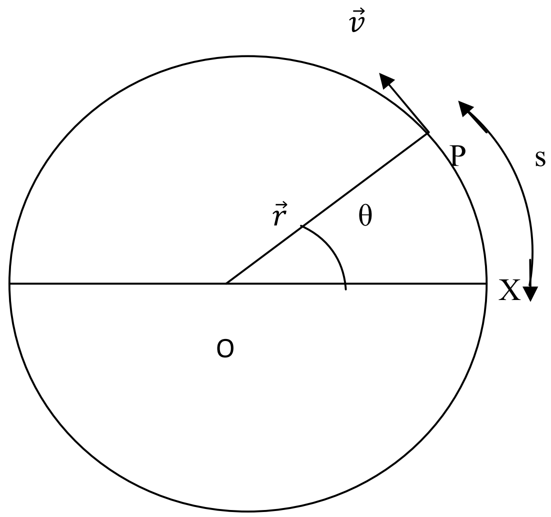

Figure 1

We know, angle = $\frac{\text{arc}}{\text{radius}}$

or $\theta = \frac{XP}{OP}$

or $\theta = \frac{s}{r}$ …..(1)

If a particle starts from position X and P_1 and P_2 be the positions of the particle at time t_1 and t_2 respectively. θ_1 and θ_2 be its angular position at time t_1 and t_2. Suppose in time interval Δt (= $t_2 - t_1$), the particle covers a distance Δs along the circular path. It revolves through the angle $\Delta\theta$ (= $\theta_2 - \theta_1$) during this time interval. The angle of revolution $\Delta\theta$ is called the 'angular displacement'

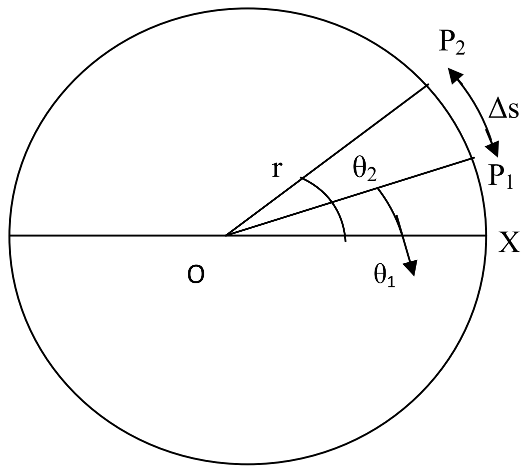

Figure 2

of the particle. If r is the radius of the circle, then the angular displacement is given by-

$\Delta\theta = \frac{\Delta s}{r}$ …..(2)

The unit of the angle or angular displacement is radian.

If $\Delta s = r$, then $\Delta\theta = \frac{1}{1} = 1$ radian

i.e. if the length of the arc of a circle is equal to the radius of the circle, then the angle subtended by the arc at the centre of the circle is 1 radian.

The whole circumference of the circle subtends an angle of 360^0 at the centre of the circle. According to the definition of radian, the angle subtended by the whole circumference ($2\pi r$) at the centre $= \frac{2\pi r}{r} = 2\pi$ radian. Hence 2π radian $= 360^0$.

6.3.2 Angular Velocity

"The time-rate of change of angular displacement is called angular velocity". It is denoted by Greek letter ω (omega) i.e.

$\omega = \frac{\theta}{t}$

or $\omega = \frac{\Delta\theta}{\Delta t}$ …..(3)

where $\Delta\theta$ is the angular displacement of the particle in time-interval Δt.

The instantaneous angular velocity of the particle is given by -

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

$$\text{or } \omega = \frac{d\theta}{dt} \qquad \text{.....(4)}$$

The unit of angular velocity is radian/sec.

In one complete revolution, the particle undergoes an angular displacement of 2π radian (or 360^0) and it takes the time T (i.e. time of period), then angular velocity of the particle is given by-

$$\omega = \frac{2\pi}{T} \qquad \text{.....(5)}$$

(since $\Delta\theta = 2\pi$, $\Delta t = T$)

If the particle makes n revolutions in 1 second, then

$$\omega = 2\pi n \qquad \text{.....(6)}$$

(since $\frac{1}{T} = n$, frequency)

The angular velocity is the characteristic of the body as a whole.

6.3.3 Angular Acceleration

If the angular velocity of a rotating body about an axis is changing with time then its motion is 'accelerated rotatory motion'. "The time-rate of change of angular velocity of a body about an axis is called angular acceleration of the body about that axis". It is denoted by α (alpha).

If the angular velocity of a body about an axis changes from ω_1 to ω_2 in time-interval $(t_2 - t_1)$, then the angular acceleration of the body about that axis is –

$$\alpha = \frac{\text{change in angular velocity}}{\text{time-interval}}$$

$$= \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

$$\text{or } \alpha = \frac{\Delta\omega}{\Delta t} \qquad \text{.....(7)}$$

The instantaneous angular acceleration of the particle is given by-

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad \text{.....(8)}$$

or $\alpha = \frac{d}{dt}(\frac{d\theta}{dt})$ (since $\omega = \frac{d\theta}{dt}$)

or $\alpha = \frac{d^2\theta}{dt^2}$ …..(9)

The angular acceleration also is the characteristic of the body as a whole. Its unit is radian/sec^2.

6.3.4 Relation between Angular Velocity and Linear Velocity

We know that $\theta = \frac{s}{r}$

or $s = r\,\theta$ …..(10)

Differentiating with respect to t, we get-

$\frac{ds}{dt} = r\frac{d\theta}{dt}$ (since r = constant)

But $\frac{ds}{dt} = v$ (linear velocity of the particle) and $\frac{d\theta}{dt} = \omega$ (angular velocity)

Therefore, $v = r\,\omega$ …..(11)

This is the relation between the magnitudes of linear velocity of a particle and the angular velocity.

In vector form, $\vec{v} = \vec{\omega} \times \vec{r}$ …..(12)

The direction $\vec{\omega}$ is always along the action of rotation, being upwards for a particle moving anti clockwise (the direction of $\vec{r} \times \vec{v}$), in fig. 2, it is normal to the plane of the paper upwards at the center O.

6.3.5 Relation between Angular Acceleration and Linear Acceleration

We know that $v = r\,\omega$

Differentiating with respect to t, we get-

$\frac{dv}{dt} = r\frac{d\omega}{dt}$ (since r = constant)

But $\frac{dv}{dt} = a_T$ (tangential component of the linear acceleration)

and $\frac{d\omega}{dt} = \alpha$ (angular acceleration of the body as a whole)

Therefore, $a_T = r\,\alpha$ …..(13)

This is the relation between tangential linear acceleration of a particle in the body at a distance r from the axis of rotation and the angular acceleration of the body.

We know that the radial (centripetal) acceleration a_R of a particle moving with velocity v in a circle of radius r is $\frac{v^2}{r}$. This can be expressed in terms of angular velocity ω of the body.

$a_R = \frac{v^2}{r} = \frac{r^2\omega^2}{r}$ (since $v = r\,\omega$)

$a_R = r\,\omega^2$ …..(14)

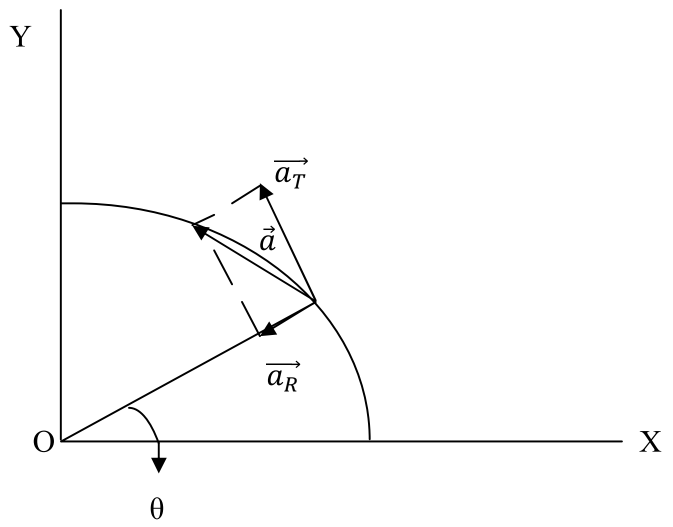

Figure 3

The resultant acceleration $\vec{a}$ of the particle is-

$\vec{a} = \overrightarrow{a_T} + \overrightarrow{a_R}$

or $a = \sqrt{a_T^2 + a_R^2}$

$= \sqrt{(r\alpha)^2 + (r\omega)^2}$

or $a = r\sqrt{\alpha^2 + \omega^4}$ …..(15)

Example 1: A car is moving with a speed of 20 m/sec on a circular track of radius 400 meter. Its speed is increasing at the rate of 4 m/sec^2. Find out the value of its acceleration.

Solution: The speed of the car moving on a circular track is increasing. Therefore, besides the centripetal (radial) acceleration a_R, the car has a tangential acceleration a_T. a_R and a_T are mutually perpendicular.

Here v = 20 m/sec. r = 400 m, a_T = 4 m/sec^2

Centripetal (radial) acceleration $a_R = \frac{v^2}{r} = \frac{(20)^2}{400}$ = 1 m/sec^2

Therefore, result acceleration $a = \sqrt{a_T^2 + a_R^2}$

$$= \sqrt{(4)^2 + (1)^2} = \sqrt{17} = 4.12 \text{ m/sec}^2$$

Example 2: The moon revolves around the earth in 2.4×10^6 sec in a circular orbit of radius 3.9 × 10^5 Km. Determine the acceleration of the moon towards the earth.

Solution: Given, r = 3.9 × 10^5 Km = 3.9 × 10^5 × 10^3 m = 3.9 × 10^8 m, T = 2.4 × 10^6 sec

Angular velocity of the moon $\omega = \frac{2\pi}{T} = \frac{2\times3.14}{2.4\times10^6}$

$$= \frac{6.28\times10^{-6}}{2.4} = 2.62 \times 10^{-6} \text{ radian/sec}$$

The acceleration of the moon towards the earth, a = r ω^2

$$= 3.9 \times 10^8 \times (2.62 \times 10^{-6})^2$$

$$= 2.68 \times 10^{-3} \text{ m/sec}^2$$

Self Assessment Question (SAQ) 1: Show that angular acceleration $\vec{\alpha}$ is perpendicular to angular velocity $\vec{\omega}$, if ω is a constant.

Self Assessment Question (SAQ) 2: Calculate the angular speed of a flywheel making 120 revolutions per minute.

Self Assessment Question (SAQ) 3: A particle is revolving round a circular path. What is the direction of acceleration of the particle?

Self Assessment Question (SAQ) 4: Two racing cars of masses m_1 and m_2 are moving in circles of radii r_1 and r_2 respectively. Their speeds are such that each of them makes a complete circle in the same time t. What is the ratio of their angular speed?

6.4 TORQUE

A force is required to produce linear acceleration in a particle. In the similar way, a torque (or moment of force) is required to produce angular acceleration in a particle about an axis.

When an external force acting on a body has a tendency to rotate the body about an axis, then the force is said to exert a 'torque' upon the body about that axis. "The torque (or moment of a force) about an axis of rotation is equal to the product of the magnitude of the force and the perpendicular distance of the line of action of the force from the axis of rotation".

In the figure 4 is shown a body which is free to rotate about an axis passing through a point O and perpendicular to the plane of the paper. Let a force F be applied on the body in the plane of the paper to rotate the body about this axis.

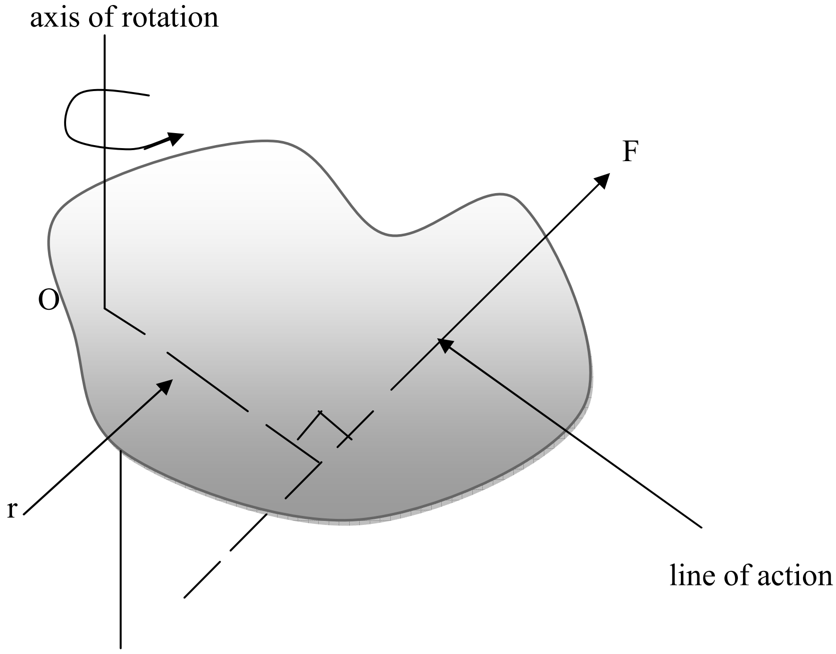

Figure 4

Let r be the perpendicular distance of the line of action of the force from the point O, then torque or moment of force F about the axis of rotation is given as-

$\tau = F \times r$ …..(16)

If the torque tends to rotate the body anticlockwise then it is taken as positive; if clockwise then negative. The unit of torque is Newton-meter and like force, it is a vector quantity.

In vector form, $\vec{\tau} = \vec{r} \times \vec{F}$ …..(17)

where $\vec{r}$ is the position vector of the point at which force acts with respect to the reference point.

Its scalar magnitude $\tau = r\ F \sin\theta$ …..(18)

If $\theta = 90^0$ i.e. r is perpendicular to the line of action of force, then

$\tau = r\ F \sin 90^0$

$= r\ F$ (maximum)

i.e. the torque is maximum when the force is applied at the right angle to $\vec{r}$. This is why in opening or closing a heavy revolving door the force is applied (by hand) at right angles to the door at its outer edge. Besides this, the torque depends also on the position of the point relative to origin at which the force is applied. If force is applied at the origin (i.e. $\vec{r}$ is zero) then no torque is produced. In this situation, the body will not rotate how-so-ever large the force may be. This is why we cannot open or close a door by applying force at the hinge.

On the contrary, greater is the distance of the line of action from origin, larger is the moment of force or torque about O; or smaller the force required to rotate the body. This fact is used in daily life. The handle revolving the grinding machine is fixed quite far from the pivot, the water-pump is fitted with a long handle and the handle of a door is fixed at a large distance from the pivot. The handle of a screw-driver is made wide because of the same reason.

6.5 MOMENT OF INERTIA

A body rotating about an axis resists any change in its rotational motion (angular velocity). On account of this property the body is said to possess a 'moment of inertia' or 'rotational inertia' about that axis. "The property of a body by virtue of which it opposes any change in its state of rotation about an axis is called the 'moment of inertia' of the body about that axis". It is denoted by I.

"The moment of inertia of a particle about an axis is given by the product of the mass of the particle and the square of the distance of the particle from the axis of rotation".

If a particle of mass m is at a distance r from an axis of rotation, its moment of inertia I about that axis is given as-

$I = mr^2$ …..(19)

Let us consider a rigid body of mass M. We have to find out the moment of inertia about a vertical axis passing through O (Figure 5). If $m_1, m_2, m_3,$ ………be the masses of the particles composing the body and $r_1, r_2, r_3,$ …………their respective distances from the axis of rotation, the moment of inertia I of the body about that axis is equal to the sum of the moments of inertia of all the particles i.e.

$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 +$……………. …..(20)

or $I = \Sigma\, mr^2$ …..(21)

Here Σ (sigma) means the sum of all terms.

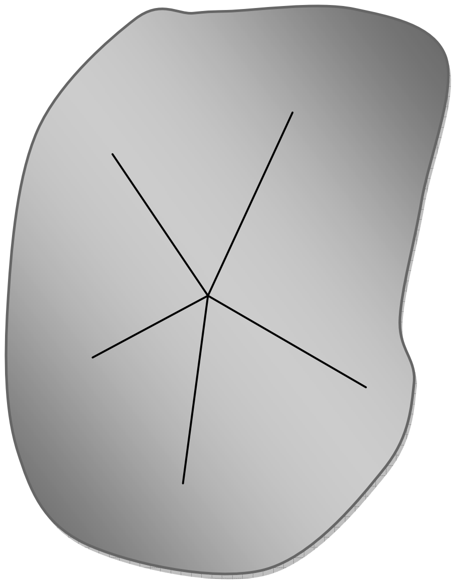

Figure 5

For a body having a continuous distribution of matter, $I = \int r^2\, dm$ …..(22)

where dm is the mass of an infinitesimally small element of the body taken at a distance r from the axis of rotation. Hence, "the moment of inertia of a rigid body about a given axis is the sum of the products of the masses of its particles by the square of their respective distances from the axis of rotation".

The unit of moment of inertia is Kg-m^2.

Obviously, the moment of inertia of a body about an axis depends not only on the mass of the body but also upon the manner in which the mass is distributed around the axis of rotation.

6.5.1 Radius of Gyration

The radius of gyration of a body about an axis of rotation is defined as the distance of a point from the axis of rotation at which whole mass of the body were assumed to be concentrated, its moment of inertia about the given axis would be the same as with its actual distribution of mass. It is usually denoted by the letter k.

If M is the total mass of the body; its moment of inertia in terms of its radius of gyration k can be written as-

$$I = M k^2 \qquad \text{.....(23)}$$

$$\text{or } \; k = \sqrt{\frac{I}{M}} \qquad \text{.....(24)}$$

Thus "the square root of the ratio of moment of inertia of the body about the given axis of rotation to its mass is called radius of gyration of the body about the given axis".

6.5.2 Physical significance of Moment of Inertia

According to Newton's first law of motion, we know that if a body is at rest or moving with a uniform speed along a straight line, then an external force is necessary to change its state. This property of bodies is called 'inertia'. Greater the mass of a body, greater is the force required to bring a change in its position of rest or in its linear velocity (i.e. to produce linear acceleration in it). In this way, the mass of a body is a measure of its inertia.

Similarly, in order to rotate a body (initially at rest) about an axis or to change the angular velocity of a rotating body (i.e. to produce an angular acceleration in it), a torque has to be applied on the body. This is described by saying that the body has a 'moment of inertia' about the axis of rotation. The greater the moment of inertia of a body about an axis, the greater is the torque required to rotate, or to stop, the body about that axis. Thus, the moment of inertia plays the same role in the rotational motion as mass plays in translational motion.

There is a difference between inertia and moment of inertia of a body. The inertia of a body depends only upon the mass of the body. But the moment of inertia of a body about an axis depends not only on the mass of the body but also upon the distribution of its mass about the axis of rotation.

6.5.3 Practical applications of Moment of Inertia

An important use of the property of moment of inertia is made in stationary engines. The torque rotating the shaft of an engine changes periodically and so the shaft cannot rotate uniformly. To keep its rotation uniform, a large heavy wheel is attached with the shaft. This wheel is called 'flywheel' and it has a large moment of inertia. As the shaft rotates, the flywheel also rotates. Due to its large moment of inertia, the flywheel (and hence the shaft) continues to rotate almost uniformly in spite of the changing torque. A small flywheel is attached to the bottom of toy-motor. The flywheel is rotated by rubbing it on the ground and the motor is left for running. Due to the moment of inertia of the flywheel, the motor continues moving for some time.

The moment of inertia plays vital role in our daily life. In cycle, rickshaw, bullock-cart, etc., the moment of inertia of the wheels is increased by concentrating most of the mass at the rim of the wheel and connecting the rim to the axle of the wheel through spokes. It is due to the large moment of inertia of the wheels that when we stop cycling, the wheels of the cycle continue rotating for some time.

6.5.4 Moment of Inertia of certain regular bodies

(i) Thin Rod: If the mass of a thin rod is M and its length is L, then the moment of inertia of the rod about an axis passing through its centre of gravity and perpendicular to its length is given by-

$I = \frac{ML^2}{12}$

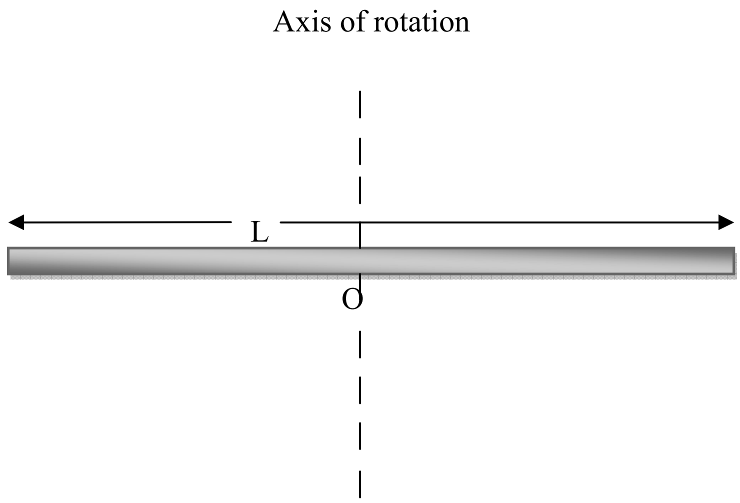

Figure 6

(ii) Rectangular Plate: The moment of inertia of a plate of mass M, length L and breadth B about an axis passing through its centre of gravity and perpendicular to its plane is given by-

$I = M\left(\frac{L^2 + B^2}{12}\right)$

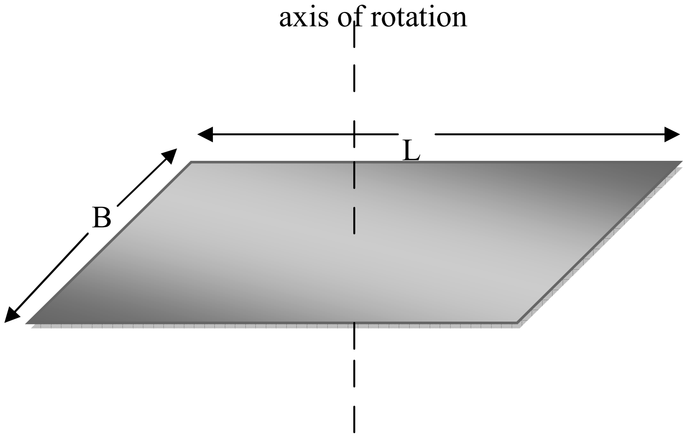

Figure 7

(iii) Ring: If the mass of a ring is M and its radius is R then its moment of inertia about its own geometrical axis is given by-

$I = MR^2$

(iv) Solid Cylindrical Rod: The moment of inertia of a solid cylinder of mass M, length L and radius R about an axis passing through its centre of gravity and perpendicular to its length is given by-

$I = M\left(\frac{L^2}{12} + \frac{R^2}{4}\right)$

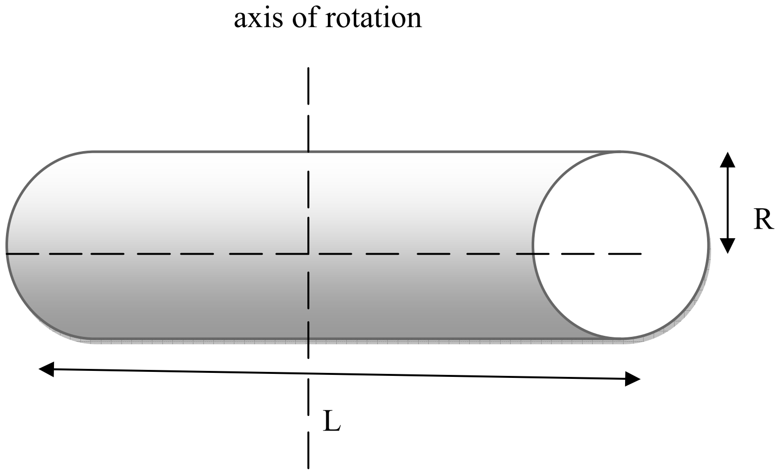

Figure 8

(v) Solid Disc: If the mass of a disc is M and the radius is R, then its moment of inertia about an axis passing through its centre of gravity and perpendicular to its plane is given by-
$I = \frac{1}{2} M R^2$

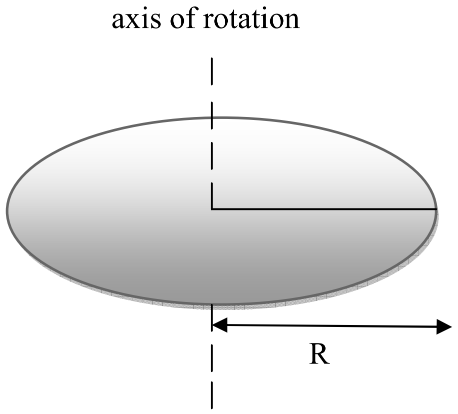

Figure 9

(vi) Solid Sphere: If the mass of a solid sphere is M and its radius is R , then its moment of inertia about a diameter is given by-

$I = \frac{2}{5} M R^2$

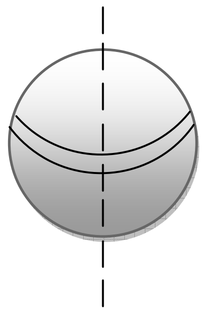

Figure 10

(vii) Spherical Shell: If the mass of a spherical shell is M and its radius is R, then its moment of inertia about a diameter is given by-

$I = \frac{2}{3} M R^2$

6.6 RELATION BETWEEN TORQUE AND MOMENT OF INERTIA

Let us consider a body acted upon by a torque τ, it is rotating about an axis passing through a fixed point O. Suppose it has a constant angular acceleration α. The angular acceleration of all the particles of the body will be the same (i.e. α), but their linear accelerations will be different. Suppose, the mass of one particle of the body is m_1 and its distance from the axis of rotation is r_1. Then the linear acceleration of this particle is given by-

$a_1 = r_1\, \alpha$; (as $v_1 = \omega r_1$; $\frac{dv_1}{dt} = r_1 \frac{d\omega}{dt}$)

If F_1 be the force acting on this particle, then

F_1 = mass × acceleration = $m_1 a_1$

$= m_1\, r_1\, \alpha$

The moment of this force about the axis of rotation passing through O = $F_1 \times r_1$

$= m_1\, r_1\, \alpha \times r_1 = m_1 r_1^2 \alpha$

Similarly, if the masses of other particles be m_2, m_3, ……. and their respective distances from the axis of rotation be r_2, r_3, ………, then the torques acting on them will be $m_2r_2^2\alpha$, $m_3r_3^2\alpha$, …… respectively. The torque τ acting on the whole body will be the sum of the torques acting on all the particles i.e.

$$\tau = m_1r_1^2\alpha + m_2r_2^2\alpha + m_3r_3^2\alpha + \ldots\ldots\ldots\ldots$$

$$= (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots\ldots\ldots)\alpha = (\Sigma\, mr^2)\,\alpha$$

But $\Sigma\, mr^2$ is the moment of inertia I of the body about the axis of rotation. Hence

$$\tau = I \times \alpha$$

torque = moment of inertia × angular acceleration

If $\alpha = 1$, then $\tau = I$ i.e. the moment of inertia of a body about an axis is equal to the torque required to produce unit angular acceleration in the body about that axis.

Example 3: A flywheel of mass 20 Kg and radius of gyration 100 cm is being acted on by a torque of 20 N-m. Determine the angular acceleration produced.

Solution: Given M = 20Kg, k = 100 cm = 1 m, τ = 20 N-m

We know $I = M\,k^2$

$$= 20 \times (1)^2 = 20 \text{ Kg m}^2$$

Again using $\tau = I \times \alpha$

or $\alpha = \frac{\tau}{I} = \frac{20}{20} = 1$ radian/sec^2

Example 4: The torque τ acting on a body of moment of inertia I about the axis of rotation is given by $\tau = (at^2 + bt + c)\,I$, where a, b, c are constants. Express angular displacement of the body (starting from rest at t = 0) as a function of t.

Solution: We know the relation between torque and moment of inertia as-

$\tau = I \times \alpha$ …..(i)

Given $\tau = (at^2 + bt + c)\,I$ …..(ii)

Comparing equations (i) and (ii), we get-

$$\alpha = at^2 + bt + c$$

or $\frac{d^2\theta}{dt^2} = at^2 + bt + c$

Integrating both sides-

$$\int \frac{d^2\theta}{dt^2}\, dt = \int (at^2 + bt + c)dt$$

or $\frac{d\theta}{dt} = \frac{at^3}{3} + \frac{bt^2}{2} + ct + A$ .....(iii)

where A is constant of integration.

At t = 0, $\frac{d\theta}{dt} = 0$

Therefore, 0 = 0 + A or A = 0

Putting for A in equation (iii) we get-

$\frac{d\theta}{dt} = \frac{at^3}{3} + \frac{bt^2}{2} + ct$ .....(iv)

Again integrating with respect to t,

$$\int \frac{d\theta}{dt}\, dt = \int \left(\frac{at^3}{3} + \frac{bt^2}{2} + ct\right) dt$$

or $\theta = \frac{at^4}{12} + \frac{bt^3}{6} + \frac{ct^2}{2} + B$ .....(v)

where B is constant of integration.

At t = 0, θ = 0, therefore from equation (v), we have-

B = 0

Therefore $\theta = \frac{at^4}{12} + \frac{bt^3}{6} + \frac{ct^2}{2}$

Self Assessment Question (SAQ) 5: Two circular discs P and Q of same mass and same thickness are made of two different metals whose densities are d_P and d_Q ($d_P > d_Q$). Their moments of inertia about the axes passing through their centres of gravity and perpendicular to their planes are I_P and I_Q. Which one has greater moment of inertia.

Self Assessment Question (SAQ) 6: A torque of 4×10^{-4} N-m is to be applied to produce an angular acceleration of 8 radian/sec^2 in a flywheel. Estimate the moment of inertia of the flywheel.

Self Assessment Question (SAQ) 7: Why is a ladder more likely to slip when you are high up on it than when you just begin to climb?

Self Assessment Question (SAQ) 8: Why is it more difficult to revolve a stone by tieing it to a longer string than by tieing it to a shorter string?

Self Assessment Question (SAQ) 9: Choose the correct option-

(i) In rotating motion, moment of inertia-

(a) imparts angular acceleration (b) imparts angular deceleration (c) aids change in rotational motion (d) opposes the change in rotational motion

(ii) In rotatory motion the physical quantity that imparts angular acceleration or deceleration is-

(a) moment of inertia (b) torque (c) force (d) angular velocity

(iii) Moment of inertia in rotational motion has its analogue in translatory motion-

(a) mass (b) torque (c) force (d) displacement

6.7 EQUATIONS OF ANGULAR MOTION

When a body rotates with constant angular acceleration, then the relations among its angular velocity, angular displacement, angular acceleration and time can be expressed by simple equations as in the case of translatory motion.

First Equation: If ω_0 be the initial angular velocity of a body rotating about a fixed axis with constant angular acceleration α, then its angular velocity after time t is given by-

$\omega = \omega_0 + \alpha\, t$

Proof: We know that angular acceleration $\alpha = \frac{d\omega}{dt}$

or $d\omega = \alpha\, dt$

Integrating both sides, we get-

$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha\, dt$

or $(\omega - \omega_0) = \alpha\,(t-0)$

or $\omega = \omega_0 + \alpha\, t$ …..(25)

Second Equation: If ω_0 be the initial angular velocity of a body rotating about a fixed axis with constant angular acceleration α, then the angle traced by the body after time t is –

$\theta = \omega_0\, t + \frac{1}{2}\,\alpha\, t^2$

Proof: We know that angular velocity $\omega = \frac{d\theta}{dt}$

But $\omega = \omega_0 + \alpha t$

Therefore, $\omega_0 + \alpha t = \frac{d\theta}{dt}$

or $\frac{d\theta}{dt} = \omega_0 + \alpha t$

$d\theta = (\omega_0 + \alpha t)\, dt$

Integrating both sides-

$\int_0^\theta d\theta = \int_0^t (\omega_0 + \alpha t)\, dt$

or $\theta = \omega_0 (t-0) + \alpha \left(\frac{t^2}{2} - 0\right)$

or $\theta = \omega_0 t + \frac{1}{2} \alpha t^2$ …..(26)

Third Equation: If ω_0 be the initial angular velocity of a body rotating about a fixed axis with constant angular acceleration α and the angular displacement in time t be θ then its angular velocity after time t is given by-

$\omega^2 = \omega_0^2 + 2\alpha\theta$

Proof: By first equation, $\omega = \omega_0 + \alpha t$

Squaring both sides-

$\omega^2 = (\omega_0 + \alpha t)^2$

or $\omega^2 = \omega_0^2 + \alpha^2 t^2 + 2\omega_0 \alpha t$

or $\omega^2 = \omega_0^2 + 2\alpha(\omega_0 t + \frac{1}{2}\alpha t^2)$

or $\omega^2 = \omega_0^2 + 2\alpha\theta$ …..(27)

(using equation 26)

6.7.1 Linear and Angular variables of a rotating body

The following table represents the variables or quantities in linear and angular motion-

Linear Motion	Angular Motion
Linear displacement s	Angular displacement θ
Linear velocity $v = \frac{ds}{dt}$	Angular velocity $\omega = \frac{d\theta}{dt}$
Linear acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$	Angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
Mass m	Moment of inertia I
Force F	Torque τ
$F = ma$	$\tau = I\ \alpha$
$W = Fs$	$W = \tau\ \theta$
$p = mv$	$J = I\ \omega$
$K = \frac{1}{2} mv^2$	$K = \frac{1}{2} I\ \omega^2$
$v = u + at$	$\omega = \omega_0 + \alpha\ t$
$s = ut + \frac{1}{2} at^2$	$\theta = \omega_0\ t + \frac{1}{2}\ \alpha\ t^2$
$v^2 = u^2 + 2\ as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$

Example 5: The wheel of a car is completing 1200 rotations in 1 minute. On pressing the accelerator of the car, the wheel makes 2400 rotations in 1 minute. Compute its angular acceleration and the angular displacement in 10 sec.

Solution: Here $\omega_0 = 2\pi n_0 = 2\pi \times \frac{1200}{60} = 40\ \pi$ radian/sec, $\omega = 2\pi n = 2\pi \times \frac{2400}{60} = 80\ \pi$ radian/sec, t = 10 sec

Using $\omega = \omega_0 + \alpha\ t$

$$80\ \pi = 40\ \pi + \alpha \times 10$$

or $\alpha = 4\ \pi$ radian/sec^2

Now using $\omega^2 = \omega_0^2 + 2\alpha\theta$

$$(80\ \ )^2 = (40\ \pi\ )^2 + 2 \times 4\ \pi\ \theta$$

or $\theta = 600\ \pi$ radian

Example 6: Moment of inertia of a ring is 3 Kg-m^2. It is rotated for 20 sec from its rest position by a torque of 6 N-m. Calculate the work done.

Solution: I = 3 Kg-m^2, $\omega_0 = 0$, t = 20 sec, τ = 6 N-m

Using $\tau = I\ \alpha$

or $\alpha = \frac{\tau}{I} = \frac{6}{3} = 2$ radian/sec^2

Using $\theta = \omega_0 t + \frac{1}{2} \alpha t^2$

$$\theta = 0\,(20) + \frac{1}{2}\,(2)\,(20)^2 = 400 \text{ radians}$$

Work done $W = \tau\, \theta$

$$= 6 \times 400 = 2400 \text{ Joule}$$

Self Assessment Question (SAQ) 10: A motor of an engine is rotating about its axis with an angular velocity of 100 revolutions/min. It comes to rest in 15 sec, after being switched off. Assuming constant angular deceleration, calculate the number of revolutions made by it before coming to rest.

6.8 ROTATIONAL KINETIC ENERGY

Let us consider a rigid body rotating about an axis with a uniform angular velocity ω. Each particle in the body has a certain kinetic energy. The angular velocity of each particle of the body will be same equal to ω but their linear velocities will be different. Suppose a particle of the body whose mass is m_1, is at a distance r_1 from the axis of rotation. Let v_1 be the linear velocity of this particle. Therefore,

$v_1 = r_1\, \omega$

Kinetic energy of this particle $K_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 (r_1\, \omega)^2 = \frac{1}{2} m_1\, r_1^2\, \omega^2$

Similarly, kinetic energies of other particles of the body having masses m_2, m_3, ………and distances r_2, r_3, ………from the axis of rotation. Then kinetic energies of these particles are given as-

$K_2 = \frac{1}{2} m_2\, r_2^2\, \omega^2$, $K_3 = \frac{1}{2} m_3\, r_3^2\, \omega^2$, ……………

The kinetic energy of the whole body will be equal to the sum of the kinetic energies of all the particles. Therefore, Kinetic energy of whole body $K = K_1 + K_2 + K_3 + \ldots\ldots\ldots\ldots..$

or $K = \frac{1}{2} m_1\, r_1^2\, \omega^2 + \frac{1}{2} m_2\, r_2^2\, \omega^2 + \frac{1}{2} m_3\, r_3^2\, \omega^2 + \ldots\ldots\ldots\ldots.$

$$= \frac{1}{2} (m_1\, r_1^2 + m_2\, r_2^2 + m_3\, r_3^2 + \ldots\ldots\ldots..)\, \omega^2$$

$$= \frac{1}{2} (\Sigma\, mr^2)\, \omega^2$$

Since $\Sigma\, mr^2 = I$, moment of inertia

$K = \frac{1}{2} I \omega^2$ …..(28)

This is the expression for rotational kinetic energy of a body. Obviously kinetic energy of rotation is equal to half the product of moment of inertia of the body and the square of the angular velocity of the body.

From above expression, $I = \frac{2K}{\omega^2}$

If $\omega = 1$, then $I = 2K$

i.e. the moment of inertia of a body rotating about an axis with unit angular velocity equals twice the kinetic energy of rotation about that axis.

If a body rotating about an axis is simultaneously moving along a straight line, then its total kinetic energy will be $(\frac{1}{2} mv^2 + \frac{1}{2} I \omega^2)$, where v is the linear velocity of the body.

6.9 ANGULAR MOMENTUM

"The moment of linear momentum of a particle rotating about an axis is called angular momentum of the particle". It is denoted by J.

If a particle be rotating about an axis of rotation, then J = linear momentum × distance

$= p \times r$

$= mv \times r$ (since $p = mv$)

or $J = mvr$

where m, v and r are the mass of the particle, linear velocity and distance of particle from axis of rotation respectively.

But $v = r\omega$, where ω is the angular velocity

Therefore, $J = m (r\omega) r = mr^2 \omega$

or $J = I \omega$ …..(29)

where $mr^2 = I$ = Moment of Inertia of particle about the axis of rotation

Let us suppose a body be rotating about an axis with an angular velocity ω. All the particles of the body will have the same angular velocity ω but different linear velocities.

Let a particle be at a distance r_1 from the axis of rotation, the linear velocity of this particle is given by-

$v_1 = r_1\omega$

If m_1 be the mass of the particle, then its linear momentum $p_1 = m_1v_1$

The moment of this momentum about the axis of rotation i.e. angular momentum of the particle J_1 = linear momentum × distance

$= p_1 \times r_1$

$= m_1v_1 \times r_1$

$= m_1(r_1\omega) \times r_1$

$= m_1r_1^2\ \omega$

Similarly, if the masses of other particles be m_2, m_3, …… and their respective distances from the axis of rotation be r_2, r_3, ……… , then the moments of their linear momenta about the axis of rotation will be $m_2r_2^2\ \omega$, $m_3r_3^2\ \omega$, ………..respectively. The sum of these moments of linear momenta of all the particles i.e. the angular momentum of the body is given by-

$J = m_1r_1^2\ \omega + m_2r_2^2\ \omega + m_3r_3^2\ \omega + \ldots\ldots\ldots$

$= (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots\ldots)\omega$

$= (\sum mr^2)\omega$

or $J = (\sum mr^2)\omega$ …..(30)

But $(\sum mr^2) = I$, the moment of inertia of the body about the axis of rotation

Therefore, angular momentum $J = I\ \omega$ …..(31)

The unit of angular momentum is Kg-m^2/sec. It is a vector quantity.

In vector form, $\vec{J} = \vec{r} \times \vec{p}$ …..(32)

or $\vec{J} = r\ p\ \sin\theta\ \hat{n}$ …..(33)

where θ is the angle between $\vec{r}$ and $\vec{p}$ and $\hat{n}$ is the unit vector perpendicular to the plane containing $\vec{r}$ and $\vec{p}$.

Magnitude of angular momentum $J = r\ p\ \sin\theta$ …..(34)

6.9.1 Relation between Torque and Angular Momentum

We know that relation for angular momentum-

$J = I\ \omega$ …..(35)

The rate of change of angular momentum-

$$\frac{\Delta J}{\Delta t} = I\ \frac{\Delta \omega}{\Delta t}$$

$$= I\ \alpha \qquad .....(36)$$

(since $\frac{\Delta\omega}{\Delta t}$ = α, angular acceleration)

But $I\ \alpha\ = \tau$ (Torque)

Therefore, $\frac{\Delta J}{\Delta t} = \tau$ .....(37)

i.e. the time-rate of change of angular momentum of a body is equal to the external torque acting upon the body. The equation (37) represents the relation between torque and angular momentum. This formula is similar to the formula $\frac{\Delta p}{\Delta t}$ = F, for linear motion.

In vector form-

$$\vec{J} = \vec{r} \times \vec{p}$$

Differentiating both sides with respect to time t, we get-

$$\frac{d\vec{J}}{dt} = \frac{d}{dt}\ (\vec{r} \times\ \vec{p})$$

$$= (\ \frac{d\vec{r}}{dt} \times \vec{p}) + (\vec{r} \times \frac{d\vec{p}}{dt})$$

$$= (\ \vec{v} \times m\vec{v}) + (\ \vec{r} \times \frac{d\vec{p}}{dt}\ ) \qquad \left(\text{since } \frac{d\vec{r}}{dt} = \vec{v} \text{ and } \vec{p} = m\vec{v}\ \right)$$

$$= m(\vec{v} \times \vec{v}) + (\ \vec{r} \times \frac{d\vec{p}}{dt}\ )$$

$$= 0 + (\ \vec{r} \times \frac{d\vec{p}}{dt}\ ) \qquad (\text{since } \vec{v} \times \vec{v} = 0)$$

or $\frac{d\vec{J}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$ .....(38)

By Newton's second law, $\frac{d\vec{p}}{dt} = \vec{F}$

Therefore, $\frac{d\vec{J}}{dt} = \vec{r} \times \vec{F}$ .....(39)

But $\vec{r} \times \vec{F} = \vec{\tau}$, the torque acting on the particle

Therefore, equation (39) becomes-

$$\frac{d\vec{J}}{dt} = \vec{\tau} \qquad \text{.....(40)}$$

i.e. the time-rate of change of angular momentum of a particle is equal to the torque acting on the particle.

6.9.2 Relation between Angular Momentum and Rotational Kinetic Energy

We know that rotational kinetic energy $K = \frac{1}{2} I\,\omega^2$

Multiplying and dividing by I in right hand side of the above expression, we get-

$$K = \frac{1}{2}\frac{I^2\omega^2}{I}$$

$$= \frac{1}{2}\frac{(I\omega)^2}{I} = \frac{1}{2}\frac{J^2}{I} \quad \text{(since } I\,\omega = J\text{)}$$

Therefore, $K = \frac{J^2}{2I}$ .....(41)

This is the relation between angular momentum and rotational kinetic energy.

Example 7: A body of mass 1 Kg is rotating on a circular path of diameter 2 m at the rate of 10 rotations in 31.4 sec. Calculate the angular momentum and rotational kinetic energy of the body.

Solution: Given, m = 1 Kg, r = 2 m/2 = 1 m, n = 10/31.4 rotations/sec

Angular velocity $\omega = 2\pi n = 2 \times 3.14 \times 10/31.4 = 2$ radian/sec

Moment of inertia $I = mr^2 = 1\,(1)^2 = 1$ Kg-m^2

Angular momentum $J = I\,\omega = 1 \times 2 = 2$ Kg-m^2/sec

Rotational kinetic energy $K = \frac{1}{2} I\,\omega^2 = \frac{1}{2} \times 1 \times (2)^2 = 2$ Joule

Self Assessment Question (SAQ) 11: A solid sphere is rolling on a table. What fraction of its total kinetic energy is rotational?

6.10 SUMMARY

In the present unit, we have studied about rotational motion, torque, moment of inertia and rotational kinetic energy of a body. We have studied about different rotational variables like angular displacement, angular velocity, angular acceleration etc. Angular velocity is defined as the time-rate of change of angular displacement while the time-rate of change of angular velocity is called as angular acceleration. We have also established the relationships between angular velocity and linear velocity as $v = r\omega$ and between angular acceleration and linear acceleration as

$a = r\alpha$. In the unit, we have studied about torque and its importance with examples. The torque or moment of force is given as the product of force applied and the perpendicular of line of action of force from the axis of rotation. In this unit, we have also covered moment of inertia and its physical significance with some practical applications. We have derived an important expression which relates moment of inertia and torque as $\tau = I\ \alpha$. Three important equations of angular (rotatory) motion have been derived in the unit. We have highlighted linear and angular variables of a rotating body. We have also established the expression of rotational kinetic energy of a body as $K = \frac{1}{2} I\ \omega^2$ and defined moment of inertia in terms of rotational kinetic energy as the twice of the kinetic energy of rotation about axis of rotation if the body is rotating with unit angular velocity. In the unit, we have also covered angular momentum, relation between torque & angular momentum and relation between angular momentum & rotational kinetic energy. We have included examples and self assessment questions (SAQs) to check your progress.

6.11 GLOSSARY

Rotational- the action of moving in a circle

Rigid- not able to be changed

Relative- considered in relation or in proportion to something else

Angular- having angles

Characteristic- a quality typical of a thing

Radial- arranged in lines coming out from a central point to the edge of a circle

Situation- a set of circumstances existing at a particular time and in a particular place

Resist- to oppose, withstand the action or effect of

Manner- a way in which something is done or happens

Distribution- the action of distributing something

Necessary- needing to be done or present, essential

Variable- often changing or likely to change , not consistent

6.12 TERMINAL QUESTIONS

1. Define angular velocity and write its unit. Give its relation with linear velocity. Show that $\vec{\omega} = (\vec{r} \times \vec{v} / r^2)$.

2. A body is moving in a circle of radius r. How much will be the distance covered by it in half the period of rotation? How much displacement?

3. How much is the angular velocity of a geostationary satellite?

4. A particle moving on a circle has a velocity of 5 m/sec and normal acceleration of 10 m/sec^2. Determine the radius of the circle.

5. The second-hand of a watch is 3 cm long. Find the linear speed of its tip.

6. Explain angular displacement. What is its unit?

7. What is angular acceleration? Establish a relation between angular acceleration and linear acceleration.

8. What is torque? Establish a relation between moment of inertia and torque.

9. The moment of inertia of a body is 6 Kg-m^2. What angular acceleration will be produced in it by applying a torque of 12 N-m on it?

10. What is radius of gyration?

11. What is moment of inertia? Give its significance.

12. A particle of mass 10 gm is rotating about an axis in a circular path of radius 20 cm. Calculate its moment of inertia.

13. Define angular momentum. Establish a relation between angular momentum and torque.

6.13 ANSWERS

Self Assessment Questions (SAQs):

1. If ω is constant, then $\frac{d\omega}{dt} = 0$

or $\frac{d}{dt}(\omega^2) = 0$

or $\frac{d}{dt}(\vec{\omega}.\vec{\omega}) = 0$

or $\frac{d\vec{\omega}}{dt}.\vec{\omega} + \vec{\omega}.\frac{d\vec{\omega}}{dt} = 0$

or $\frac{d\vec{\omega}}{dt}.\vec{\omega} + \frac{d\vec{\omega}}{dt}.\vec{\omega} = 0$ (since $\vec{A}.\vec{B} = \vec{B}.\vec{A}$)

or $2\frac{d\vec{\omega}}{dt}.\vec{\omega} = 0$

or $2\,\vec{\alpha}.\vec{\omega} = 0$ (since $\frac{d\vec{\omega}}{dt} = \vec{\alpha}$)

or $\vec{\alpha}.\vec{\omega} = 0$

i.e. $\vec{\alpha}$ is perpendicular to $\vec{\omega}$.

2. Here n = 120 revolution/minute = 120/60 = 2 revolution/sec

Angular speed $\omega = 2\pi n = 2 \times 3.14 \times 2 = 12.56$ radian/sec

3. along the radius

4. angular speed $\omega = \frac{2\pi}{T}$. Here time taken to complete the circle (T) is same. Therefore, both cars have same (equal) angular speed. Hence the ratio of their angular speed = 1: 1

5. Let the mass of each disc be m and their radii be r_P and r_Q.

Moment of inertia of disc P, $I_P = \frac{1}{2} m r_P^2$

Moment of inertia of disc Q, $I_Q = \frac{1}{2} m r_Q^2$

Therefore, $\frac{I_P}{I_Q} = \frac{r_P^2}{r_Q^2}$ …..(i)

Let t be the thickness of each disc, then mass of disc P, $m = \pi r_P^2 \times t \times d_P$

mass of disc Q, $m = \pi r_Q^2 \times t \times d_Q$

Therefore, $\pi r_P^2 \times t \times d_P = \pi r_Q^2 \times t \times d_Q$

or $\frac{r_P^2}{r_Q^2} = \frac{d_Q}{d_P}$

From equation (1), we have $\frac{I_P}{I_Q} = \frac{d_Q}{d_P}$

Since $d_P > d_Q$, therefore $I_Q > I_P$

i.e. the circular disc Q has greater moment of inertia.

6. Given, $\tau = 4 \times 10^{-4}$ N-m, $\alpha = 8$ radian/sec^2

We know, $\tau = I\,\alpha$

or $I = \frac{\tau}{\alpha} = \frac{4\times10^{-4}}{8} = 5 \times 10^{-5}$ Kg-m^2

7. When a man is high up on the ladder the moment of force tending to rotate the ladder about its base increases. When he just begins to climb, this moment is small and is insufficient to cause slipping.

8. We know, $\tau = I\,\alpha$ and $I = mr^2$

If r is greater i.e. the stone is farther from the axis of rotation, greater is moment of inertia and hence more torque is required to cause same angular acceleration with longer string than that with shorter string. Evidently, it will be more difficult to revolve a stone tied to a longer string than that tied with a shorter string.

9. (i) (d), (ii) (b), (iii) (a)

10. $\omega_0 = 2\,\pi\, n = 2\,\pi \times \frac{100}{60} = 10\,\pi/3$ radian/sec, t = 15 sec, $\omega = 0$

Using $\omega = \omega_0 + \alpha\, t$

$0 = 10\,\pi/3 + \alpha\,(15)$ or $\alpha = -2\,\pi/9$ radian/sec^2

Using $\theta = \omega_0\, t + \frac{1}{2}\,\alpha\, t^2$

$\theta = 10\,\pi/3 \times 15 + \frac{1}{2}\,(-2\,\pi/9)(15)^2 = 25\,\pi$ radian

$2\,\pi$ radian angular displacement = 1 revolution

Therefore $25\,\pi$ radian angular displacement $= \frac{1}{2\pi} \times 25\,\pi = 12.5$ revolution

11. Moment of inertia of solid sphere $I = \frac{2}{5} MR^2$

Where M is the mass of the solid sphere and R the radius of sphere.

Rotational kinetic energy of sphere $K_{rot} = \frac{1}{2} I\,\omega^2 = \frac{1}{2}\,(\frac{2}{5} MR^2)\,\omega^2 = \frac{1}{5} MR^2\,\omega^2$

Translatory kinetic energy of sphere $K_{trans} = \frac{1}{2} Mv^2 = \frac{1}{2} M\,(R\omega)^2$ (since $v = r\omega$)

$= \frac{1}{2} M\,R^2\,\omega^2$

Total kinetic energy of solid sphere $K = K_{rot} + K_{trans} = \frac{1}{5} MR^2\,\omega^2 + \frac{1}{2} M\,R^2\,\omega^2 = \frac{7}{10} M\,R^2\,\omega^2$

Thus fraction of rotational kinetic energy $\frac{K_{rot}}{K} = \frac{\frac{1}{5}MR^2\omega^2}{\frac{7}{10}MR^2\omega^2} = \frac{2}{7}$

Terminal Questions:

1. we know that $\vec{v} = \vec{w} \times \vec{r}$

$\vec{r} \times \vec{v} = \vec{r} \times (\vec{w} \times \vec{r})$

$= (\vec{r}.\vec{r})\,\vec{w} - (\vec{r}.\vec{w})\,\vec{r}$

$= r^2\vec{w} - 0$ (as $\vec{r} \perp \vec{\omega}$)

Hence $\vec{\omega} = (\vec{r} \times \vec{v})/ r^2$

2. Distance covered in half the period of rotation = half of the circumference of the circle

$= ½ (2\pi r) = \pi$ r

Displacement = direct distance between initial and final point = diameter of the circle = 2 r

3. Time period of revolution of geostationary satellite T = 24 hours

Therefore, its angular velocity $\omega = \frac{2\pi}{T} = \frac{2\pi}{24} = \frac{\pi}{12}$ radian/hour

4. Given v= 5 m/sec, a = 10 m/sec^2

$a = \frac{v^2}{r}$ or $r = \frac{v^2}{a} = \frac{10^2}{10} = 2.5$ m

5. Here r = 3 cm = 0.03 m, For second-hand T = 60 sec

$\omega = \frac{2\pi}{T} = \frac{2\pi}{60} = \frac{\pi}{30}$ radian/sec

$v = r\,\omega = 0.03 \times \frac{\pi}{30} = 3.14 \times 10^{-3}$ m/sec

9. Given I = 6 Kg-m^2, τ = 12 N-m

$\tau = I\,\alpha$ or $\alpha = \frac{\tau}{I} = \frac{12}{6} = 2$ radian/sec^2

12. Given m= 10 gm = 10^{-2} Kg, r = 20 cm = 0.2 m

Moment of inertia $I = mr^2 = 10^{-2} \times (0.2)^2 = 4 \times 10^{-4}$ Kg-m^2

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UNIT 7: MOTION OF CHARGED PARTICLE IN CROSSED ELECTRIC AND MAGNETIC FIELDS

STRUCTURE:

7.1 INTRODUCTION

In the previous units, you have studied some important concepts and issues of mechanics with reference to translatory and rotatory motion of a particle. In this unit, we shall study and analyze the dynamics of charged particle. We shall recall electric and magnetic forces acting on charged particles. In the present unit, we shall deal with the motion of charged particle in uniform constant electric field, alternating electric field, uniform magnetic field and crossed electric and magnetic fields. You shall also learn about velocity selector which is an important application of combination of a uniform crossed electric and magnetic fields.

7.2 OBJECTIVES

After studying this unit, you should be able to-

- understand electric and magnetic forces
- compute electric and magnetic forces acting on charged particles
- solve problems based on dynamics of charged particles
- understand the motion of charged particle in electric and magnetic fields

7.3 ELECTRIC FORCE

We know that two like charges repel each other and two unlike charges attract each other. Thus it is evident that a force acts between two charges. This force is known as 'electric force'. The electric force between like charges is repulsive and that between unlike charges is attractive. If electric charges be placed in vacuum even then electric force acts between them.

In 1785, Coulomb, on the basis of experiments, gave a law regarding the force acting between two charges which is known as Coulomb's law. According to this law, two stationary point charges q_1 and q_2 repel or attract each other with a force F which is directly proportional to the product of the charges and inversely proportional to the square of the distance r between them i.e.

$$F \propto \frac{q_1 q_2}{r^2}$$

$$\text{or} \quad F = k\frac{q_1 q_2}{r^2} \qquad \text{.....(1)}$$

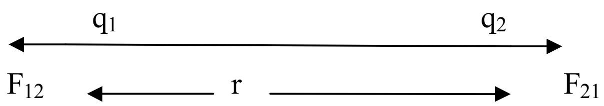

Figure 1

where k is a proportionality constant.

If the charges are placed in vacuum (or air), then k = $\frac{1}{4\pi\varepsilon_0}$

Therefore, $F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$ …..(2)

The force F acts along the line joining the charges and the force on q_1 is equal and opposite to that on q_2.

The value of $\frac{1}{4\pi\varepsilon_0}$ is equal to 9×10^9 N-m^2/C^2

ε_0 (epsilon zero) is called 'permittivity of free space' and its value is equal to 8.85×10^{-12} C^2/N-m^2.

In vector form, $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$ …..(3)

where $\hat{r}$ is the unit vector along r.

When the surrounding medium is not vacuum (or air) but some insulating material (such as wax, paper, glass etc.) then the force between the charges is given by-

$F = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2}$ …..(4)

where ε is called the absolute permittivity of the material medium and $\varepsilon = \varepsilon_0 K$, where K is called the dielectric constant or relative permittivity or specific inductive capacity of the material and the material is called dielectric. The expression (4) can be written as-

$F = \frac{1}{4\pi\varepsilon_0 K}\frac{q_1 q_2}{r^2}$ …..(5)

The space surrounding an electric charge in which another charge experiences a force of attraction or repulsion is called the electric field of that charge. The electric field strength E at any point in the electric field is defined as the force experienced by unit charge when placed at that point i.e.

$E = \frac{F}{q}$ …..(6)

where F is the force experienced by the charge q.

In vector form, $\vec{E} = \frac{\vec{F}}{q}$ …..(7)

7.4 MAGNETIC FORCE

The region near a magnet, where a magnetic needle experiences a torque and rests in a definite direction, is called 'magnetic field'. A charge moving in a magnetic field experiences a deflecting force. Of course, if a charge moving through a point experiences a deflecting force, then a magnetic field is said to exist at that point. This field is represented by a vector quantity $\vec{B}$, called magnetic field or magnetic induction.

Let us consider a charged particle of charge q which is moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$, then the magnetic force acting on that charged particle is given by Lorentz force as-

$$\vec{F} = q(\vec{v} \times \vec{B}) \qquad \text{.....(8)}$$

The magnitude of Lorentz force is given as-

$$F = qvB \sin\theta \qquad \text{.....(9)}$$

where θ is the angle between velocity and magnetic field.

The magnetic field itself is caused by moving charges or current loops and the magnetic force arises due to interaction between one set of moving charges with the other set of moving charges.

Both the electric and magnetic forces are combinedly known as electromagnetic forces. If a particle carrying a charge q is moving in space where both an electric field $\vec{E}$ and a magnetic field $\vec{B}$ are present, then the force on the particle will be given by-

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B})$$

or $$\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})] \qquad \text{.....(10)}$$

7.5 MOTION OF CHARGED PARTICLE IN UNIFORM CONSTANT ELECTRIC FIELD

Let us consider a charged particle of charge q in a uniform constant electric field $\vec{E}$, then the electric force experienced by charged particle is given by-

$$\vec{F} = q\vec{E} \qquad \text{.....(11)}$$

If m is the mass of the particle then the acceleration produced in the particle due to this force is-

$$\vec{F} = m\vec{a}$$

or $$\vec{F} = m\frac{d^2\vec{r}}{dt^2} \qquad \text{.....(12)}$$

where $\vec{r}$ is the displacement of the particle at any time t.

From equations (11) and (12), we have-

$$q\vec{E} = m\frac{d^2\vec{r}}{dt^2}$$

or $\frac{d^2\vec{r}}{dt^2} = \frac{q}{m}\vec{E}$

i.e. $\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{q}{m}\vec{E}$ …..(13)

The above expression gives the acceleration of the charge in the field $\vec{E}$.

Integrating equation (13), we get-

$$\int \frac{d^2\vec{r}}{dt^2} dt = \int \frac{q\vec{E}}{m} dt + A$$

or $\frac{d\vec{r}}{dt} = \frac{q\vec{E}}{m} t + A$ …..(14)

where A is constant of integration.

Let initially at time t = 0, $\vec{v} = \vec{v_0}$ i.e. $\frac{d\vec{r}}{dt} = \vec{v_0}$ then from equation (14), we have-

$\vec{v_0} = A$ or $A = \vec{v_0}$

Putting for A in equation (14), we get-

$\frac{d\vec{r}}{dt} = \frac{q\vec{E}}{m} t + \vec{v_0}$ …..(15)

or $\vec{v} = \frac{q\vec{E}}{m} t + \vec{v_0}$

Obviously, the velocity $(\frac{d\vec{r}}{dt})$ of charged particle is linear function of time or in other words, velocity increases linearly with time.

Again integrating equation (15), we get-

$$\int \frac{d\vec{r}}{dt} dt = \int \frac{q\vec{E}}{m} t\, dt + \int \vec{v_0}\, dt$$

or $\vec{r} = \frac{q\vec{E}}{m}\frac{t^2}{2} + \vec{v_0}\, t + B$ …..(16)

where B is constant of integration.

Initially, at t = 0, $\vec{r} = \overrightarrow{r_0}$, then from equation (16), we have-

$\overrightarrow{r_0} = B$ or $B = \overrightarrow{r_0}$

Putting for B in equation (16), we get-

$$\vec{r} = \frac{q\vec{E}}{m}\frac{t^2}{2} + \overrightarrow{v_0}\ t + \overrightarrow{r_0} \qquad \text{.....(17)}$$

Obviously, the displacement ($\vec{r}$) varies in a quadratic manner with time.

If the particle starts from rest from origin of coordinate axes, then $\overrightarrow{v_0} = 0$ and $\overrightarrow{r_0} = 0$, then equation (17) takes the form-

$$\vec{r} = \frac{q\vec{E}}{m}\frac{t^2}{2} \qquad \text{.....(18)}$$

Now let us discuss two cases of particular interest.

Case I : When the field is applied along the direction of motion of particle i.e. longitudinal electric field

Let the electric field acts along X-axis and the charged particle be moving in the same direction, then at time t = 0, $\overrightarrow{v_0} = \overrightarrow{v_{0x}}$ and $x = x_0$

Since the force due to electric field acts along X-direction only, the acceleration of the particle will also be in the X-direction and there will be no acceleration along Y and Z directions. The Cartesian components of equation (13) can be written as-

$a_x = \frac{qE_x}{m}$ .....(a)

$a_y = 0$ .....(b)

$a_z = 0$ .....(c) .....(19)

Integrating equation 19(a), and using $v_0 = v_{0x}$ at t = 0, the velocity of particle along X-direction is given as-

$$v_x = v_{0x} + a_x t \qquad \text{.....(20)}$$

Integrating equation (20) and using $x = x_0$ at t = 0, the displacement is given by-

$$x = x_0 + v_{0x}\, t + \frac{1}{2} a_x t^2 \qquad \text{.....(21)}$$

where $a_x = \frac{qE_x}{m}$ is given by equation 8(a).

Case II: When electric field is applied perpendicular to the direction of motion of particle i.e. transverse electric field

Let us consider that the charged particle is moving with an initial uniform velocity v_{0x} in the X-direction and the constant electric field be applied in the Y-direction, then at t = 0, we have-

$v_x = v_{0x}$, $v_y = v_z = 0$ and $x = y = z = 0$

Also $E_x = E_z = 0$ and $E_y = E$

Since there is no force along X and Z-directions, there will be no acceleration along X and Z-directions. The Cartesian components of acceleration are, therefore,

$a_x = a_z = 0$ and $a_y = \frac{qE_y}{m} = \frac{qE}{m}$

But the particle has an initial velocity in X-direction, hence it will continue to move in X-direction with the same velocity. Velocity components are, therefore,

$v_x = v_{0x}$, $v_y = \frac{qE}{m} t$ and $v_z = 0$

and hence the displacements are given by-

$x = v_{0x}\, t$, $y = \frac{1}{2}\frac{qE}{m} t^2$ and $z = 0$

Eliminating t between the equations of x and y, we have for the actual path traversed by the particle-

$$y = \frac{1}{2}\frac{qE}{m}\left(\frac{x}{v_{0x}}\right)^2 = \frac{qE}{2mv_{0x}^2} x^2$$

$$\text{or} \quad x^2 = \frac{2mv_{0x}^2}{qE} y \qquad \text{.....(22)}$$

Obviously, this equation represents a parabola. Thus the path traversed by a charged particle, moving with a uniform initial velocity and subjected to a constant transverse electric field, is a parabola as shown in figure.

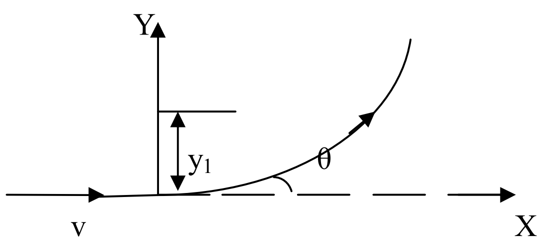

Figure 2

The transverse displacement that the charged particle suffers during passage through the plates of length l is given by-

$$y_1 = \frac{1}{2} at^2 = \frac{1}{2}\frac{qE}{m}\left(\frac{l}{v_{0x}}\right)^2$$

$$= \frac{qEl^2}{2mv_{0x}^2}$$

After emerging from the plates, the charged particle experiences no force and hence moves along a straight line. Its direction of travel after emerging from the electric field will be inclined to the original direction of travel (X-axis) by an angle by –

$$\tan\theta = \frac{v_y}{v_x}$$

$$= \frac{qE}{mv_{0x}} t = \frac{qEl}{mv_{0x}^2}$$

7.6 MOTION OF CHARGED PARTICLE IN ALTERNATING ELECTRIC FIELD

In the case of alternating field, the electromotive force (EMF) and current vary continuously from a maximum in one direction through zero to a maximum in the opposite direction. Let us consider a charged particle moving in alternating electric field of the form-

$$E = E_0 \sin\omega t \qquad \text{.....(23)}$$

where E_0 is the peak value of E and $\omega = 2\pi f$ is the angular frequency of the electric field vector, f being the frequency of field.

We know that the acceleration of charged particle in field $\vec{E}$ is-

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{q}{m}\vec{E}$$

$$\text{or } \vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{q}{m} E_0 \sin\omega t \qquad \text{.....(24)}$$

This expression represents that the acceleration of the charged particle varies (figure 4) in a sinusoidal way like field.

Integrating equation (24), we get-

$$\int \frac{d^2\vec{r}}{dt^2} dt = \int \frac{q}{m} E_0 \sin\omega t \, dt$$

$$\text{or } \frac{d\vec{r}}{dt} = -\frac{q}{m\omega} E_0 \cos\omega t + A$$

where A is constant of integration.

Let initially at t = 0, the particle is at rest i.e. $\frac{d\vec{r}}{dt} = 0$

Therefore, from above expression, we have-

$0 = -\frac{q}{m\omega} E_0 \cos \omega(0) + A$

or $A = \frac{qE_0}{m\omega}$

Therefore, the velocity of the particle at time t is given by-

$\frac{d\vec{r}}{dt} = -\frac{q}{m\omega} E_0 \cos \omega t + \frac{qE_0}{m\omega}$

or $\vec{v} = \frac{d\vec{r}}{dt} = \frac{qE_0}{m\omega}(1 - \cos \omega t)$ …..(25)

Obviously, the velocity is a periodic function of time.

$\vec{v} = 0$ if $1 - \cos \omega t = 0$

or $\cos \omega t = 1$

or $\omega t = \pm 2n\pi$, where n = 0, 1, 2, …………

$\vec{v}$ is maximum when $(1 - \cos \omega t)$ is maximum or $\cos \omega t$ is minimum i.e.

$\cos \omega t = -1$

or $\omega t = (2n + 1)\pi$, where n = 0, 1, 2, …………

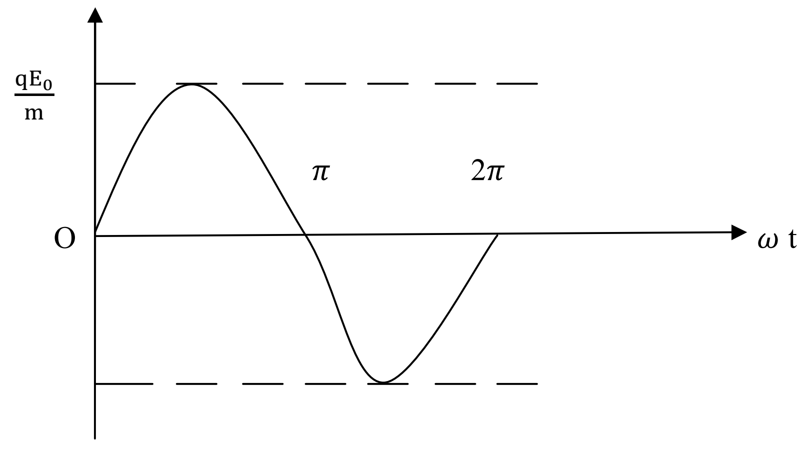

Figure (4)

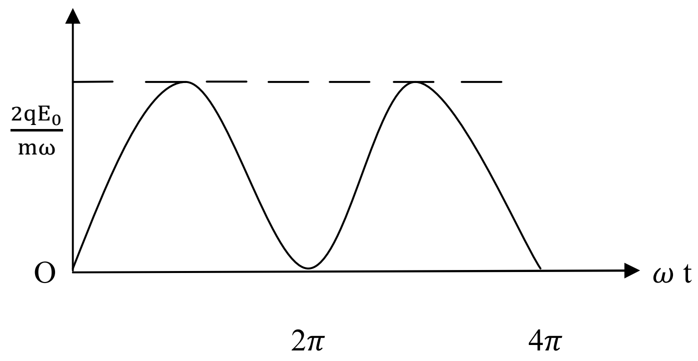

Figure (5)

Integrating equation (25), we have-

$\int \frac{d\vec{r}}{dt} dt = \int \frac{qE_0}{m\omega} (1 - \cos \omega t)\, dt$

or $\vec{r} = \frac{qE_0}{m\omega} t - \frac{qE_0}{m\omega^2} \sin \omega t + B$

where B is another constant of integration.

If initially, at t = 0, r = 0, then

$0 = \frac{qE_0}{m\omega} (0) - \frac{qE_0}{m\omega^2} \sin \omega(0) + B$

or 0 = 0 – 0 + B

or B = 0

Therefore, the above expression becomes-

$\vec{r} = \frac{qE_0}{m\omega} t - \frac{qE_0}{m\omega^2} \sin \omega t + 0$

or $\vec{r} = \frac{qE_0}{m\omega} t - \frac{qE_0}{m\omega^2} \sin \omega t$

or $\vec{r} = \frac{qE_0}{m\omega^2} [\sin \omega t - \omega\, t]$ …..(26)

The magnitude of first terms in equation (26) varies in a sinusoidal way while the magnitude of second term varies linearly with time. The resultant effect is that the motion consists of an oscillation superimposed upon a constant drift velocity $\frac{qE_0}{m\omega}$ as shown in figure (6).

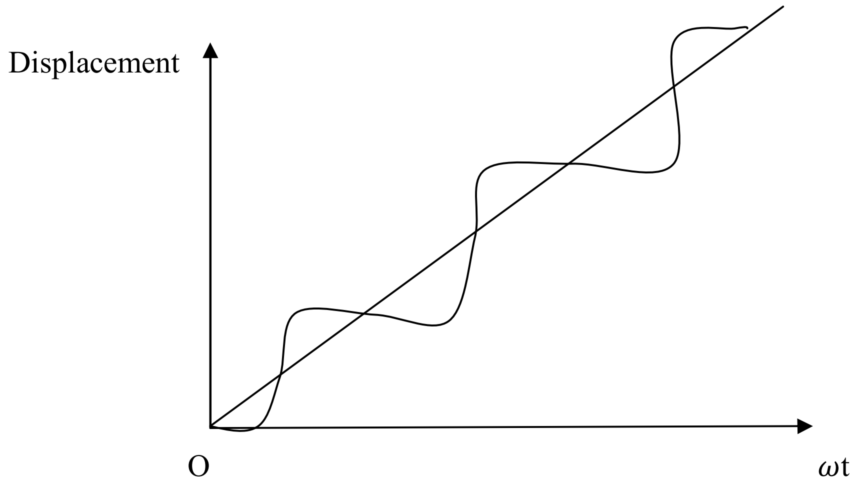

Figure (6)

The figure (4) shows the variation of acceleration with ωt. It is a pure sine curve. Figure (5) represents the variation of velocity with ωt and indicates that velocity is always positive in the direction of E_0. It lies between 0 and $\frac{2qE_0}{m\omega}$ and is never negative.

Example 1: Calculate the Coulombian force acting between a proton and an electron separated by 8×10^{-16} meter.

Solution: Here, q_1 = charge on proton = 1.6×10^{-19} coulomb

q_2 = charge on electron = 1.6×10^{-19} coulomb, r = 8×10^{-16} meter

Coulombian force acting between a proton and a electron $F = \frac{1}{4\pi\varepsilon}\frac{q_1 q_2}{r^2}$

$$= 9\times10^9\times \frac{1.6\times10^{-19}\times1.6\times10^{-19}}{(8\times10^{-16})^2}$$

= 360 Newton (attractive in nature)

Example 2: An electron is moving northwards with a velocity of 3×10^7 m/sec in a uniform magnetic field of 10 Tesla directed eastward. Find the magnitude of the magnetic force acting on electron.

Solution: Here, $v = 3\times 10^7$ m/sec, B = 10 Tesla, $\theta = 90^0$, $q = 1.6\times 10^{-19}$ coulomb

Magnetic force on electron $F = qvB \sin\theta$

$$= 1.6\times 10^{-19}\times 3\times10^7\times10 \sin 90^0 = 4.8 \times 10^{-11} \text{ Newton}$$

Example 3: An electron has an initial velocity $10^4\ \hat{i}$ m/sec and its position is $\hat{j}$ m. Compute its position and velocity after 10^{-8} sec, in an electric field of 300 $\hat{i}$ volt/m.

Solution: $\vec{v_0} = 10^4\ \hat{i}$ m/sec, $q = 1.6 \times 10^{-19}$ Coulomb, $m= 9 \times 10^{-31}$ Kg, $\vec{E} = 300\ \hat{i}$, $t = 10^{-8}$ sec

We know, $\vec{v} = \frac{q\vec{E}}{m} t + \vec{v_0}$

$$= -\frac{1.6\times10^{-19}}{9\times10^{-31}}(300\ \hat{i}) \times 10^{-8} + 10^4\ \hat{i}$$

$$= 5.2\times 10^4\ \hat{i} \text{ m/sec}$$

Position $\vec{r} = \frac{q\vec{E}}{m}\frac{t^2}{2} + \vec{v_0}\ t + \vec{r_0}$

$$= \frac{-1.6\times10^{-19}\times300\ \hat{i}(10^{-8})^2}{9\times10^{-31}\times2} + 10^4\ \hat{i}\times 10^{-8} + \hat{j}$$

$$= (-24 \times 10^{-4}\ \hat{i} + \hat{j})\ \text{m}$$

Self Assessment Question (SAQ) 1: Calculate the force between two protons when the distance between them is 0.4×10^{-14} m.

Self Assessment Question (SAQ) 2: An electron enters along the electric line of force. Discuss its motion.

Self Assessment Question (SAQ) 3: A proton moving in a straight line enters a strong magnetic field along the field direction. How will its path and velocity change?

Self Assessment Question (SAQ) 4: A proton enters in a magnetic field of strength B Tesla with speed v, parallel to the direction of magnetic field. What will be the magnetic force acting on proton?

7.7 MOTION OF CHARGED PARTICLE IN UNIFORM MAGNETIC FIELD

We know that a charged particle moving in a magnetic field experiences a magnetic force (Lorentz force) which is given by-

$$\vec{F} = q(\vec{v} \times \vec{B}) \qquad .....(27)$$

The magnitude of this force can be written as-

$F = qvB \sin\theta$

where θ is the angle between the direction of velocity and magnetic field. Now let us discuss the following two cases-

Case I: When the charged particle enters in the magnetic field parallel to the direction of the magnetic field.

Obviously, the angle between the direction of velocity and magnetic field $\theta = 0^0$

Therefore the force acting on the particle $F = qvB \sin 0^0$

$$= qvB\ (0) = 0$$

i.e. if the charged particle initially moving parallel to a magnetic field will continue to move with initial constant speed on a straight line.

Case II: When the charged particle enters in the magnetic field at right angles to the direction of the field.

Let there be a uniform magnetic field $\vec{B}$ perpendicular to the plane of the paper directed vertically downward, indicated by symbols x in the figure (7). Suppose a charged particle of mass m and carrying a charge +q enters in the field at a point O with velocity $\vec{v}$ directed perpendicular to the field $\vec{B}$. The magnetic force acting on the charged particle is given by-

$\vec{F} = q(\vec{v} \times \vec{B})$

The magnitude of this force can be written as-

$F = qvB \sin\theta$

where θ is the angle between the direction of velocity and magnetic field.

Since $\vec{v}$ is perpendicular to $\vec{B}$ i.e. $\theta = 90^0$

Therefore, the magnitude of magnetic force acting on particle $F = qvB$ …..(28)

Thus, the particle at O is acted upon by a force of magnitude qvB , lying in the plane of the paper and directed upward (right -hand screw rule). Since the magnetic force is perpendicular to the velocity, it does not change the magnitude of the velocity (i.e. speed); it changes only the direction of the velocity. Thus, the particle moves under a force whose magnitude remains constant but the direction changes continuously and is always perpendicular to the velocity. It, therefore, describes an anticlockwise circular path with constant speed v, the force $\vec{F}$ working as the centripetal force i.e. The magnetic force acting on the particle provides necessary centripetal force for the circular motion. Thus,

$$F = qvB = \frac{mv^2}{r} \qquad .....(29)$$

where r is the radius of the circle.

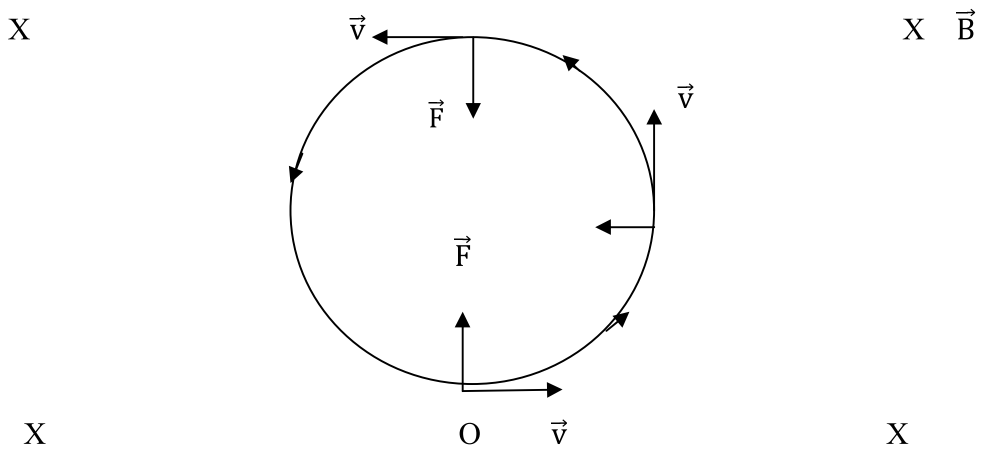

Figure (7)

The momentum of the particle is seen to be-

$$mv = qBr \qquad .....(30)$$

and the radius of the circular path as-

$$r = \frac{mv}{qB} \qquad .....(31)$$

Obviously, $r \propto mv$ i.e. the radius of the path is proportional to the momentum mv of the charged particle. The radius r is sometimes called the gyro radius or cyclotron radius.

The angular velocity ω is given by from equation (31) as-

$$\omega = \frac{v}{r} = \frac{qB}{m} \qquad .....(32)$$

The particle traverses a distance 2π r in one revolution. The time-period T is given by-

$$T = \frac{2\pi r}{v}$$

Putting for r from equation (31), we get-

$$T = \frac{2\pi m}{qB} \qquad \text{.....(33)}$$

and the frequency n is given by-

$$n = \frac{1}{T} = \frac{qB}{2\pi m} \qquad \text{.....(34)}$$

Equations (33) and (34) show that the time-period, or the frequency, of the particle is independent of the speed v of the particle. If the speed of the particle increases, its radius also increases so that the time taken to complete one revolution remains same. The frequency n is the characteristic frequency of the particle in the field and is sometimes called gyro frequency or cyclotron frequency of the particle in the field because the particles circulate at this frequency in the cyclotron.

If the particle is negatively charged, then the force at O would have been a downward force and the particle would have described a clockwise circle.

If two identical charged particles enter in the field with different speeds v_1 and v_2 ($v_2 > v_1$), then they move along circles of smaller and larger radii respectively (Figure 8).

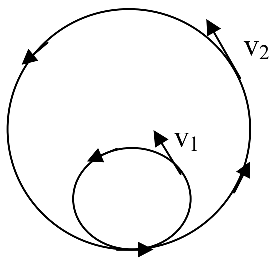

Figure (8)

Case III: When the charged particle enters in the magnetic field obliquely.

Now suppose the velocity $\vec{v}$ of the charged particle entering in the magnetic field $\vec{B}$, instead of being perpendicular to $\vec{B}$, makes an angle θ with it, then $\vec{v}$ may be resolved into components-

(i) $v_{\parallel} = v \cos\theta$ parallel to $\vec{B}$ and

(ii) $v_{\perp} = v \sin\theta$ perpendicular to $\vec{B}$

The parallel component $v_{\parallel} = v \cos\theta$ remains unaffected by the field and hence the charged particle continues to move along the field with a speed of $v \cos\theta$ i.e. this component $v_{\parallel} = v \cos\theta$

gives a linear path to the particle while the perpendicular component $v_\perp = v \sin\theta$ gives a circular path to the particle.

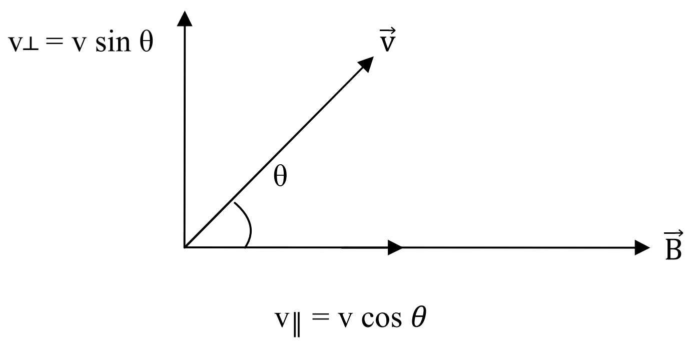

Figure (9)

The radius of the circular path is given in accordance with equation (31) as-

$$r = \frac{mv_\perp}{qB} = \frac{mv \sin\theta}{qB} \qquad .....(35)$$

The time taken by the particle for one revolution is given by-

$$T = \frac{2\pi r}{v_\perp} = \frac{2\pi r}{v \sin\theta}$$

Putting for r from equation (35), we get-

$$T = \frac{2\pi}{v \sin\theta}\left(\frac{mv \sin\theta}{qB}\right)$$

$$\text{or } T = \frac{2\pi m}{qB} \qquad .....(36)$$

Obviously, the charged particle possesses two concurrent motions- one circular motion with a constant speed $v \sin\theta$ in a plane perpendicular to the direction of magnetic field and another linear motion with constant speed $v \cos\theta$ along the direction of the field. On account of these concurrent motions, the resultant path of the charged particle will be a helix as shown in figure (10).

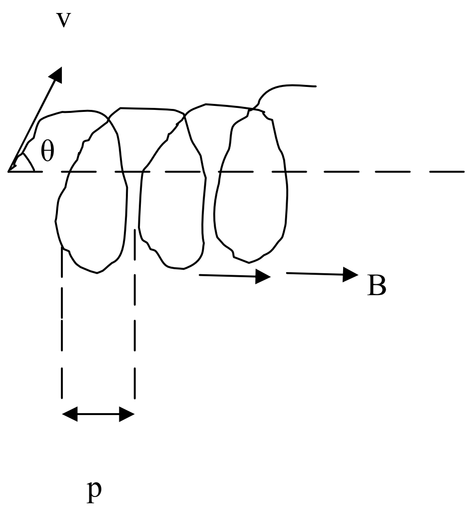

Figure (10)

The linear distance travelled by the particle in the direction of the magnetic field in one complete revolution i.e. in time T is called the 'pitch' p of the helix and is given by-

$$p = v_{\parallel} \times T = v\cos\theta \times \frac{2\pi m}{qB}$$

$$= \frac{2\pi m v\cos\theta}{qB} \qquad \text{.....(37)}$$

This helical path traced by a charged particle in a magnetic field is utilized for focusing the beam in cathode ray tubes and electron microscope etc.

Example 4: An electron is moving with a speed of 3×10^7 m/sec along a circular path in a magnetic field of 0.5 N $A^{-1}m^{-1}$. Calculate the radius of the circle.

Solution: $v = 3 \times 10^7$ m/sec, $B = 0.5$ N $A^{-1}m^{-1}$, $q = 1.6\times10^{-19}$ C, mass of electron $m = 9\times10^{-31}$ Kg

We know, $r = \frac{mv}{qB}$

$$= \frac{9\times10^{-31}\times3\times10^{7}}{1.6\times10^{-19}\times0.5} = 0.34 \times 10^{-3} \text{ m} = 0.34 \text{ mm}$$

Self Assessment Question (SAQ) 5: Choose the correct option-

(i) Magnetic field do not interact with-

(a) electric charges in motion (b) electric charges at rest (c) permanent magnets in motion (d) permanent magnets at rest

(ii) A charged particle enters at 45^0 to the magnetic field. Its path becomes-

(a) straight line (b) elliptical (c) circular (d) helical

Self Assessment Question (SAQ) 6: A charged particle moving in a straight line enters in a strong magnetic field along the field direction. How will it's path and velocity change?

7.8 CHARGED PARTICLE IN CROSSED ELECTRIC AND MAGNETIC FIELDS

Let us discuss the motion of charged particles in crossed electric and magnetic fields. Let us consider a particle of charge q and mass m which is situated in an electric field acting along Y-axis and a magnetic field perpendicular to it along Z-axis. Thus,

$\vec{E} = \hat{j}\,E$ and $\vec{B} = \hat{k}\,B$

Let the particle starts from rest in such crossed electric and magnetic fields, then initially it will be accelerated in the direction of electric field. As the velocity of particle increases, it experiences an increasing force due to magnetic field and hence its path is deflected more and more from vertical in a direction perpendicular to the plane containing the electric and magnetic field vectors. This bending of the path of charged particle continues till it reaches a point on X-axis where its velocity is once again zero and it again moves up towards the Y-axis and repeats a similar motion. Thus the path of the charged particle is a series of half loops as shown in figure (11).

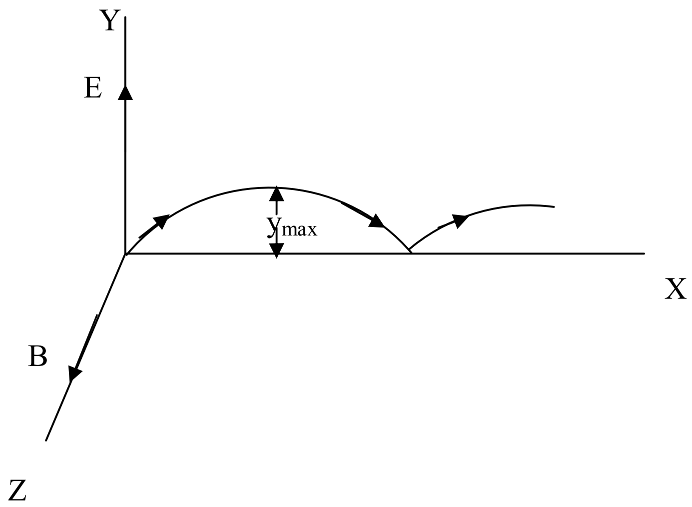

Figure (11)

Let the velocity of the charged particle at any instant is $\vec{v}$.

The net force acting on the particle = force due to electrostatic field + force due to magneto statics field

or $\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B})$

$= q\,[\vec{E} + (\vec{v} \times \vec{B})]$

$= q\,[\hat{j}\,E + \{(\hat{i}\,v_x + \hat{j}\,v_y + \hat{k}\,v_z) \times \hat{k}\,B\}]$ (since $\vec{v} = \hat{i}\,v_x + \hat{j}\,v_y + \hat{k}\,v_z$)

$= \hat{j}\,q\,E - \hat{j}\,v_x\,q\,B + \hat{i}\,v_y\,qB$ (using the properties of cross product of vectors)

or $\hat{i}\,F_X + \hat{j}\,F_y + \hat{k}\,F_z = \hat{i}\,qBv_y + \hat{j}\,(q\,E - qBv_x)$

Now comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on both sides, we get-

$$F_X = m\,\frac{dv_x}{dt} = qBv_y \quad \text{.....(38)}$$

$$F_y = m\,\frac{dv_y}{dt} = q\,E - qBv_x \quad \text{.....(39)}$$

$$F_z = m\,\frac{dv_z}{dt} = 0 \quad \text{.....(40)}$$

Differentiating equation (39) with respect to time t and substituting for $\frac{dv_x}{dt}$, we get-

$$m\frac{d^2v_y}{dt^2} = 0 - qB\left(\frac{qB}{m}\right)v_y$$

or $\frac{d^2v_y}{dt^2} = -\left(\frac{qB}{m}\right)^2 v_y$

$$= -\,\omega^2\,v_y \quad \text{.....(41)}$$

where $\omega = \frac{qB}{m}$ .....(42)

Let the solution of equation (41) be-

$$v_y = A_1 \cos \omega t + A_2 \sin \omega t \quad \text{.....(43)}$$

But since at t = 0, $v_y = 0$, therefore

$A_1 = 0$

Also at t = 0, $v_x = 0$; from equation (39), we have-

$$\frac{dv_y}{dt} = \frac{qE}{m} \quad \text{.....(44)}$$

From equation (43), we have-

$\frac{dv_y}{dt}$ = - A_1 ω sin ωt + A_2 ω cos ωt

At t = 0, $\frac{dv_y}{dt}$ = - A_1 ω sin ω(0) + A_2 ω cos ω(0)

or $\frac{dv_y}{dt} = A_2 \omega$ …..(45)

Comparing equations (44) and (45), we get-

$A_2 \omega = \frac{qE}{m} = k$ (say) …..(46)

or $A_2 = \frac{k}{\omega}$

Putting for A_1 and A_2 in equation (43), we get-

$v_y = (0)\cos \omega t + \frac{k}{\omega}\sin \omega t$

or $v_y = \frac{k}{\omega}\sin \omega t$ …..(47)

Now from equation (39), we get-

$\frac{dv_y}{dt} = \frac{qE}{m} - \frac{qB}{m} v_x$

$= k - \omega v_x$ [using equations (42) and (46)]

Therefore, $v_x = \frac{k}{\omega} - \frac{1}{\omega}\frac{dv_y}{dt}$

$= \frac{k}{\omega} - \frac{1}{\omega}\frac{d}{dt}(\frac{k}{\omega}\sin \omega t)$ Putting for v_y from equation (47)

$= \frac{k}{\omega} - \frac{1}{\omega}\frac{k}{\omega}\omega \cos \omega t$

$= \frac{k}{\omega}(1 - \cos \omega t)$ …..(48)

From equation (48), we have-

$v_x = \frac{k}{\omega}(1 - \cos \omega t)$

or $\frac{dx}{dt} = \frac{k}{\omega}(1 - \cos \omega t)$

Integrating with respect to time t, we get-

$\int \frac{dx}{dt}\, dt = \int \frac{k}{\omega}\,(1 - \cos\omega t)dt$

or $x = \frac{k}{\omega}\left(t- \frac{\sin\omega t}{\omega}\right) + k_1$ …..(49)

where k_1 is constant of integration.

Similarly, from equation (47), we have-

$v_y = \frac{k}{\omega}\sin\omega t$

or $\frac{dy}{dt} = \frac{k}{\omega}\sin\omega t$

Integrating with respect to time t, we get-

$\int \frac{dy}{dt} dt = \int \frac{k}{\omega}\sin\omega t\, dt$

or $y = - \frac{k}{\omega}\cos\omega t \left(\frac{1}{\omega}\right)$

or $y = - \frac{k}{\omega^2}\cos\omega t + k_2$ …..(50)

where k_2 is constant of integration.

On applying initial conditions, at t = 0; x = 0, y = 0, we have-

$k_1 = 0$ and $k_2 = \frac{k}{\omega^2}$

Putting $k_1 = 0$ in equation (49) and $k_2 = \frac{k}{\omega^2}$ in equation (50), we get-

$x = \frac{k}{\omega^2}(\omega t - \sin\omega t)$

or $x = \frac{mE}{qB^2}(\omega t - \sin\omega t)$ …..(51)

and $y = \frac{k}{\omega^2}(1 - \cos\omega t)$

or $y = \frac{mE}{qB^2}(1 - \cos\omega t)$ …..(52)

The above equations (51) and (52) are of cycloid which is the curve traced out by a point on the circumference of a circle rolling along a straight line and is sketched in figure (11). The maximum displacement y_{max} in the Y-direction is obtained for cos ωt = -1 i.e. ωt = π.

Therefore, $y_{max} = \frac{2mE}{qB^2}$ …..(53)

The phenomenon of crossed electric and magnetic fields has been applied by J.J. Thomson to determine the ratio of $\frac{q}{m}$ for an electron.

7.8.1 Velocity Selector

The combination of a uniform electric and magnetic fields, perpendicular to each other can be used as a velocity selector. Let us suppose that the electric field $\vec{E}$ and magnetic field $\vec{B}$ are acting along Y and Z-axis respectively and a charged particle passes through this in a direction perpendicular to either of the fields (i.e. at the origin $v_y = v_z = 0$ and $v_x \neq 0$), then for a positively charged particle the force qE due to the electric field is along + Y –direction and magnetic force qBv_x is along – Y-direction. The two forces can just balance each other if-

$$qBv_x = qE \qquad .....(54)$$

$$\text{or} \quad v_x = \frac{E}{B} \qquad .....(55)$$

In this condition, the charged particle will experience no force and hence will continue to move without any deflection along a straight line with a constant velocity. Thus, for given B and E, only one value of v_x will satisfy equation (55) and hence the particles with velocities other than v_x will be deviated. Therefore, by an adjustment of B or E, it is possible to select particles of any preferred velocity. Thus, the system behaves as a filter for charged particles since it allows only those particles whose velocity is $\frac{E}{B}$. Such a system is known as velocity selector. It is obvious that the process of velocity selection is independent of the charge and mass of the particle and all the particles of speed $v_x = \frac{E}{B}$, irrespective of their masses and charges pass undeflected.

The figure (12) shows the arrangement used in a velocity selector. A fine beam of charged particles, emerging from slit S_1 passes through a region of mutually perpendicular electric and magnetic fields. The magnetic field is generally applied with the help of a electromagnet in a

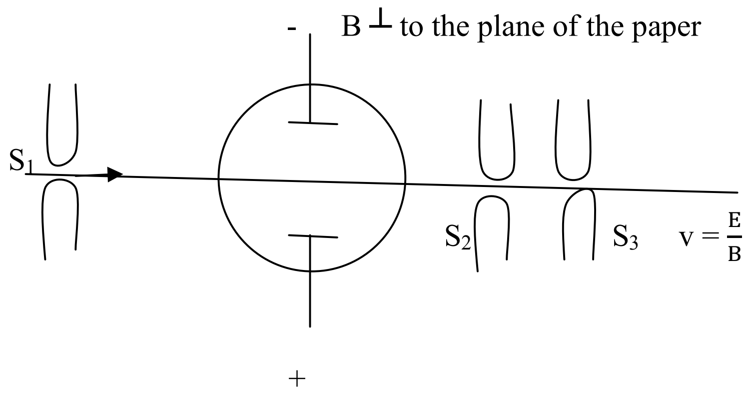

Figure (12)

direction perpendicular to the plane of the paper. The electric field is applied by placing two plates parallel to X-axis and applying a high potential difference across them. Thus the electric field is parallel to the plane of the paper i.e. perpendicular to the magnetic field and perpendicular to the direction of the beam. In this way, the two fields-electric and magnetic fields, exert force on the charged particles in opposite directions and if they are adjusted in such a way that the forces due to two fields mutually cancel, then the charged particles with speed $\frac{E}{B}$ will pass undeflected. If a series of slits S_2 and S_3 are used to collect these undeflected particles, we get a beam of particles all having the same speed. Only the particles with velocity $\frac{E}{B}$ will come out on the other side of the slits S_2 and S_3 . Other particles will be deflected to the sides and will strike against the plates and stopped there.

Example 5: A positively charged ion beam moving in the X- direction enters in a region in which there is an electric field E_y = 4000 volt/cm and the magnetic field $B_z = 4 \times 10^{-2}$ Tesla. Deduce the speed of those ions which may pass undeflected through the region.

Solution: Given E_y = 4000 volt/cm = 4×10^5 volt/m, $B_z = 4 \times 10^{-2}$ Tesla

The ion beam will pass undeflected if the force due to electric field is just balanced by the force due to magnetic field i.e.

$q E_y = q B_z v_x$

or $v_x = \frac{E_y}{B_z} = \frac{4\times 10^5}{4 \times 10-2} = 10^7$ m/sec

Self Assessment Question (SAQ) 7:A magnetic field of 0.1 Tesla is crossed with an electric field between two parallel plates 3 cm apart and having a potential difference of 500 volts. What must be the velocity of charged particles which can go undeflected through the fields perpendicular to both the fields?

7.9 SUMMARY

In this unit, we have studied and analyzed the dynamics of charged particle. We had a glimpse of electric and magnetic fields and forces acted on charged particle due to these fields. In the present unit, we have studied the motion of charged particle in uniform constant electric field and alternating electric field. We have seen that in uniform constant electric field, the velocity of charged particle increases linearly with time while the displacement varies in a quadratic manner with time. We have learnt that in transverse electric field, the trajectory of charged particle is parabola. We have also studied the motion of charged particle in uniform constant magnetic field and found that if a charged particle initially moving parallel to a magnetic field continues to move with initial constant speed on a straight line while on entering in magnetic field at right angle, the path of the particle is circular. When the particle enters in the magnetic field obliquely, the resultant path of the charged particle is a helix. We have also discussed the motion of charged particle in crossed electric and magnetic fields. In the unit, we have also studied velocity selector. We have included some examples and self assessment questions(SAQs) to check your progress.

7.10 GLOSSARY

Recall- remember something

Deal- distribute something

Analyze- examine something in detail so as to explain it

Deflect- turn aside from a straight course

Transverse- placed or extending across something

Longitudinal- extending lengthwise

Trajectory- the path followed by a moving object

Uniform- not varying, the same in all cases and at all times

Oblique- at an angle, slanting

Emerging- rising

Characteristic- a quality typical of a thing

Instant- happening immediately

7.11 TERMINAL QUESTIONS

1. Explain electric and magnetic forces. Give the expressions for electric and magnetic forces.

2. The dielectric constant of water is 81. Calculate its absolute permittivity.

3. Two charges in air experience a Coulomb force of 10N. If the space between them is filled with a medium of dielectric constant K = 4, what will be the new force?

4. An electron moving with velocity 5×10^7 m/sec enters in a magnetic field of 1 Tesla at an angle of 30^0 to the field. Determine the force acting on the electron.

5. Discuss the motion of a charged particle in uniform constant electric field. Establish the expression for transverse displacement suffered by particle in passing through the uniform electric field.

6. Describe the motion of a charged particle in a uniform magnetic field. Show that for a charged particle moving perpendicular to a magnetic field, the radius of the path is proportional to the momentum and inversely proportional to the specific mass of the particle. Also prove that the frequency of revolution of the particle is independent of its speed.

7. A beam of protons enters in a uniform magnetic field of 0.3 Tesla with a velocity of 4×10^5 m/sec at an angle of 60^0 to the field. Find the radius of the helical path taken by the beam. Also find the pitch of the helix . Mass of proton is 1.67×10^{-27} Kg

8. Does a neutron moving in a magnetic field experience force?

7.12 ANSWERS

Self Assessment Questions (SAQs):

1. $q_1 = q_2 = 1.6 \times 10^{-19}$ C, $r = 0.4\times 10^{-14}$ m $= 4\times 10^{-15}$ m

Force $F = = \frac{1}{4\pi\varepsilon_0}\frac{q_1\, q_2}{r^2} = 9\times10^9\times \frac{1.6\times10^{-19}\times1.6\times10^{-19}}{(4\times10^{-15})^2} = 14.4$ N (repulsive)

2. The electron moving along the line of force is decelerated, therefore its velocity decreases, becomes zero and then it retraces its path (if the field has sufficient length) and finally comes out of the field, with the same initial speed.

3. When a charged particle enters in the magnetic field parallel to field direction (i.e. $\theta = 0^0$), the force experienced by particle $F = qvB \sin\theta = qvB \sin 0^0 = 0$

Therefore, the path and velocity of particle will remain unaffected.

4. $\theta = 0^0$, $F = qvB \sin\theta = qvB \sin 0^0 = 0$

5. (i) (b), (ii) (d)

6. When a charged particle enters in the magnetic field parallel to the field direction, then $\theta = 0^0$

The force experienced by the particle $F = qvB \sin\theta = qvB \sin 0^0 = 0$

Therefore, the path and velocity of particle will remain unaffected.

7. B = 0.1 Tesla, d = 3 cm = 0.03 m, V = 500 volts

We know that, $E = \frac{V}{d} = \frac{500}{0.03} = 3\times10^4$ volt/metre

The charged particles can go undeflected when they have a speed v given by-

$qE = qvB$ or $v = \frac{E}{B}$

Therefore, $v = \frac{3\times10^4}{0.1} = 3\times10^5$ m/sec

Terminal Questions:

2. $K = 81$, $\varepsilon = \varepsilon_0 K = 8.85\times10^{-12}\times81 = 7.17\times10^{-10}$ $C^2/N\text{-}m^2$

3. In air, $F_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$ …..(i)

In medium, $F_2 = \frac{1}{4\pi\varepsilon_0 K}\frac{q_1 q_2}{r^2}$ …..(ii)

Dividing (i) by (ii)-

$\frac{F_1}{F_2} = K$ or $F_2 = \frac{F_1}{K} = \frac{10}{4} = 2.5$ N

4. $v = 5\times 10^7$ m/sec, $B = 1$ Tesla, $\theta = 30^0$, $q = 1.6\times10^{-19}$ C

$F = qvB\sin\theta = 1.6\times10^{-19}\times 5\times 10^7\times1\times \sin 30^0 = 1.6\times10^{-19}\times 5\times 10^7\times \frac{1}{2} = 4\times10^{-12}$ N

7. $v = 4\times 10^5$ m/sec, $\theta = 60^0$, $B = 0.3$ Tesla, $m = 1.67\times 10^{-27}$ Kg, $q = 1.6\times 10^{-19}$ C

Resolving the velocity of the proton parallel and perpendicular to the magnetic field-

$v_{\parallel} = v\cos 60^0 = 4\times 10^5\times \frac{1}{2} = 2\times10^5$ m/sec

$v_{\perp} = v\sin 60^0 = 4\times 10^5\times \frac{\sqrt{3}}{2} = 3.46\times10^5$ m/sec

$v_{\perp}$ makes the proton move on a circular path, whereas $v_{\parallel}$ takes it along the field B. Hence the path of the proton is a helix whose radius is given by-

$r = \frac{mv_{\perp}}{qB} = \frac{1.67\times 10^{-27}\times3.46\times10^5}{1.6\times 10^{-19}\times0.3} = 12\times10^{-3}$ m

Time period $T = \frac{2\pi r}{v_{\perp}} = \frac{2\times3.14\times12\times10^{-3}}{3.46\times10^5} = 21.75\times10^{-8}$ sec

Pitch of the helix $p = v_{\parallel}\times T = 2\times10^5\times21.75\times10^{-8} = 4.35\times10^{-2}$ m

8. No, since magnetic force $F = qvB\sin\theta$. Neutron is electrically neutral i.e. $q = 0$ for neutron. Therefore, $F = (0)vB\sin\theta = 0$

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UNIT 8: MOMENT OF INERTIA

Structure

8.1 INTRODUCTION

In the units 5 and 6, you have studied about angular momentum and its conservation, moment of inertia and its physical significance, radius of gyration, equations of angular motion and rotational kinetic energy. In this unit, we shall study some more important concepts of motion and laws of rotational motion. We know that, to find the moment of inertia of a body about a given axis, all that we have to do is to find the sum $\Sigma\ mr^2$ for all particles making up the body by integration or other means. The calculations to find the moment of inertia can be made shorter by the help of some important theorems. In this unit, we shall also study those theorems (theorems of parallel and perpendicular axes).

8.2 OBJECTIVES

After studying this unit, you should be able to-

- Solve problems based on equations of motion
- apply equations of motion
- understand laws of rotational motion
- apply theorems of parallel and perpendicular axes.

8.3 EQUATIONS OF MOTION

If an object is moving in a straight line under a constant acceleration, then relations among its velocity, displacement, time and acceleration can be represented by equations. These equations are called 'equations of motion'.

Let us consider that a body starts with an initial velocity 'u' and has a constant acceleration 'a'. Suppose it covers a distance's' in time't' and its velocity becomes 'v'. Then the relations among u, a, t, s and v can be represented by three equations.

First Equation: We know that linear acceleration $a = \frac{dv}{dt}$

or $dv = a\ dt$

Integrating both sides, we get-

$$\int_u^v dv = \int_0^t a\ dt$$

or $(v - u) = a\ (t - 0)$

or $v = u + at$ .....(1)

This is known as first equation of motion.

Second Equation: We know that linear velocity $v = \frac{ds}{dt}$

But v = u + at

Therefore, $u + at = \frac{ds}{dt}$

or $\frac{ds}{dt} = u + at$

or ds = (u + at) dt

Integrating both sides, we get-

$\int_0^s ds = \int_0^t (u + at)dt$

or $s = u (t – 0) + a (\frac{t^2}{2} - 0)$

or $s = ut + \frac{1}{2} a t^2$ .....(2)

This is called second equation of motion.

Third Equation: By first equation, v = u + at

Squaring both sides-

$v^2 = (u + at)^2$

or $v^2 = u^2 + a^2t^2 + 2uat$

$= u^2 + 2a (ut + \frac{1}{2} a t^2)$

or $v^2 = u^2 + 2as$ …..(3)

(using equation 2)

The above equation (3) is known as third equation of motion.

Example 1: A train starting from rest is accelerated by 0.5 m/sec^2 for 10 sec. Calculate its final velocity after 10 sec. Also calculate the distance travelled by train in 10 sec.

Solution: Here, u = 0, a = 0.5 m/sec^2, t = 10 sec

Using first equation of motion v = u + at

$v = 0 + 0.5 \times 10 = 5$ m/sec

Using second equation of motion $s = ut + \frac{1}{2} a t^2$

$s = 0\,(10) + \frac{1}{2} \times 0.5\,(10)^2$

$= \frac{1}{2} \times 0.5 \times 100 = 25$ m

Thus the velocity of train after 10 sec is 5 m/sec and the distance travelled is 25 m.

Example 2: A car is accelerated from 8 m/sec to 14 m/sec in 3 sec. What is the acceleration of car ?

Solution: Here, u = 8 m/sec, v = 14 m/sec, t = 3 sec

Using $v = u + at$

$14 = 8 + a \times 3$

$3a = 14 - 8$

$= 6$

$a = 2$ m/sec^2

Self Assessment Question (SAQ) 1: A car is moving with a constant speed of 30 Km/hr. Calculate the distance travelled by car in 1 hr.

Self Assessment Question (SAQ) 2: A particle is shot with constant speed 6×10^6 m/sec in an electric field which produces an acceleration of 1.26×10^{14} m/sec^2 directed opposite to the initial velocity. How far does the particle travel before coming to rest?

Self Assessment Question (SAQ) 3: The initial velocity of a particle is 'u' (at t = 0) and the acceleration is given by 'at^2'. Which of the following relations is valid?

(a) $v = u + at$ (b) $v = u + \frac{at^3}{3}$ (c) $v^2 = u + at^3$ (d) $v = u + \frac{at^3}{2}$

8.4 NEWTON'S LAWS OF ROTATIONAL MOTION

As we know, there are Newton's three laws of translational motion, similarly we have the following three laws of rotational motion-

First Law: Unless an external torque is applied to it, the state of rest or uniform rotational motion of a body about its fixed axis of rotation remains unaltered.

Second Law: The rate of change of angular momentum (or the rate of change of rotation) of a body about a fixed axis of rotation is directly proportional to the torque applied and takes place in the direction of the torque.

Third Law: When a torque is applied by one body on another, an equal and opposite torque is applied by the latter on the former about the same axis of rotation.

8.5 GENERAL THEOREMS ON MOMENT OF INERTIA

There are two important theorems on moment of inertia which, in some cases, facilitate the moment of inertia of a body to be determined about an axis, if its moment of inertia about some other axis be known. These theorems are theorem of parallel axes and theorem of perpendicular axes. Let us discuss these theorems.

8.5.1 Theorem of Parallel Axes

It states that the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass plus the product of the mass of the body and the square of the perpendicular distance between the two axes.

If 'I_{cm}' be the moment of inertia of a body about a parallel axis through its centre of mass, 'M' be the mass of the body and 'r' be the perpendicular distance between two axes, then moment of inertia of the body $I = I_{cm} + Mr^2$

This is the "theorem of parallel axes".

Proof: Let us consider a plane lamina with 'C' as centre of mass. Let 'I' be its moment of inertia about an axis PQ in its plane and I_{cm} the moment of inertia about a parallel axis RS passing through C. Let the distance between RS and PQ be 'r'.

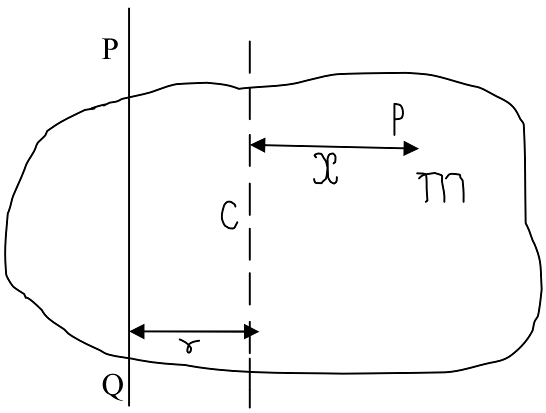


Figure 1

Let us consider a particle P of mass m at a distance x from RS. Its distance from PQ is (r + x) and its moment of inertia about it is $m(r + x)^2$. Therefore, the moment of inertia of the lamina about PQ is given by-

$I = \Sigma\, m(r + x)^2$

$= \Sigma\, m(r^2 + x^2 + 2rx)$

$= \Sigma\, mr^2 + \Sigma\, mx^2 + \Sigma\, 2mrx$

or $I = r^2\, \Sigma\, m + \Sigma\, mx^2 + 2r\, \Sigma\, mx$ …..(4)

(since r is constant)

But $\Sigma\, mx^2 = I_{cm}$, where I_{cm} is the moment of inertia of the lamina about RS, $r^2\, \Sigma\, m = r^2 M$ where M is the total mass of the lamina and $\Sigma\, mx = 0$ because the sum of the moments of all the mass particles of a body about an axis through the centre of mass of the body is zero. Hence, the equation (4) becomes

$I = r^2 M + I_{cm} + 0$

or $I = I_{cm} + M r^2$ …..(5)

It may be seen clearly from equation (5) that the moment of inertia of a body about an axis through the centre of mass is the least. The moment of inertia of the body about an axis not passing through the centre of mass is always greater than its moment of inertia about a parallel axis passing through the centre of mass of the body.

8.5.2 Theorem of Perpendicular Axes

According to this theorem, the moment of inertia of a uniform plane lamina about an axis perpendicular to its plane is equal to the sum of its moments of inertia about any two mutually perpendicular axes in its plane intersecting on the first axis.

If I_x and I_y be the moments of inertia of a plane lamina about two mutually perpendicular axes OX and OY in the plane of the lamina and I_z be its moment of inertia about an axis OZ, passing through the point of intersection O and perpendicular to the plane of the lamina, then

$I_z = I_x + I_y$

This is the “theorem of perpendicular axes”.

Proof: Let OZ be the axis perpendicular to the plane of the lamina about which the moment of inertia is to be taken. Let OX and OY be two mutually perpendicular axes in the plane of the lamina and intersecting on OZ.

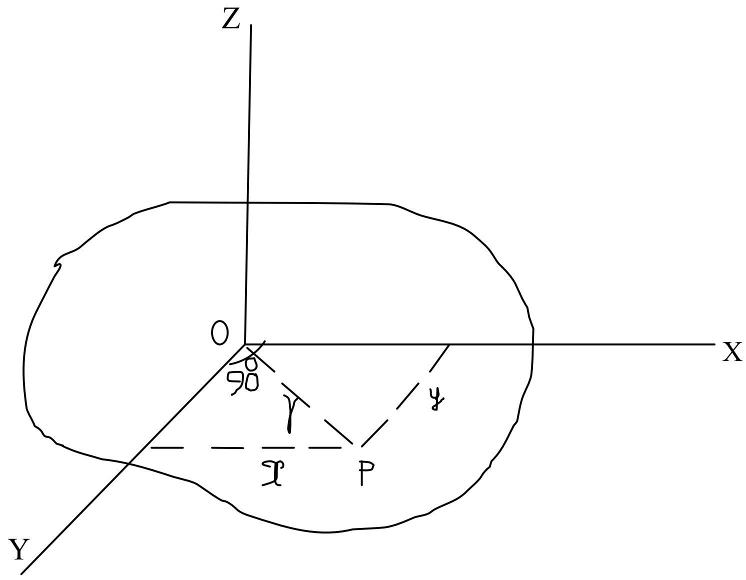


Figure 2

Let us consider a particle P of mass 'm' at a distance of 'r' from OZ. The moment of inertia of this particle about OZ is mr^2. Therefore, the moment of inertia I_z of the whole lamina about OZ is $I_z = \Sigma\, mr^2$

But $r^2 = x^2 + y^2$, where x and y are the distances of P from OY and OX respectively.

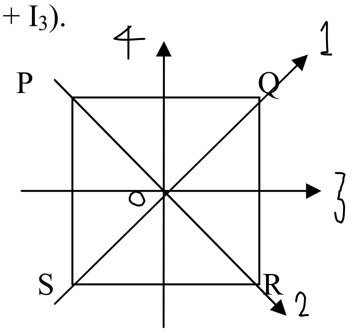
Therefore, $I_z = \Sigma\, m(x^2 + y^2) = \Sigma\, mx^2 + \Sigma\, my^2$

But $\Sigma\, mx^2$ is the moment of inertia I_y of the lamina about OY and $\Sigma\, my^2$ is the moment of inertia I_x of the lamina about OX.

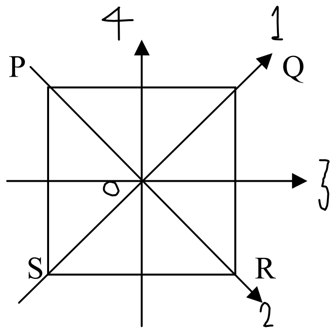
Therefore, $I_z = I_y + I_x$

or $I_z = I_x + I_y$

Example 3: Show that the moment of inertia I of a thin square plate PQRS (Figure) of uniform thickness about an axis passing through the centre O and perpendicular to the plane of the plate is $(I_1 + I_2)$ or $(I_3 + I_4)$ or $(I_1 + I_3)$.



Solution:



Let I_1, I_2, I_3 and I_4 are the moments of inertia about axes 1, 2, 3 and 4 respectively which are in the plane of the plate.

By the theorem of perpendicular axes, we have-

$I = I_1 + I_2 = I_3 + I_4$

By symmetry of square plate, $I_1 = I_2$ and $I_3 = I_4$

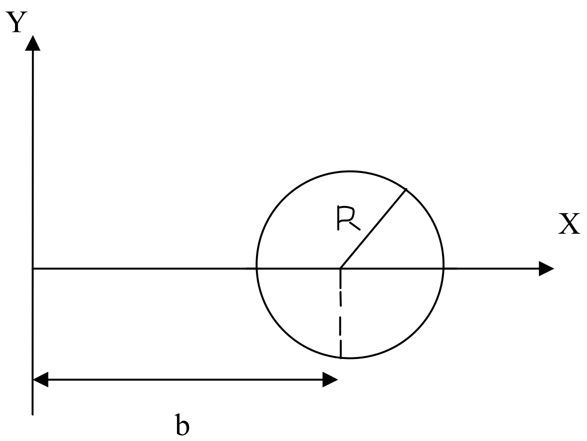
Therefore, $I = 2I_1 = 2\ I_3$

or $I_1 = I_3$

Thus $I_1 = I_2 = I_3 = I_4$

and $I = I_1 + I_2 = I_1 + I_3$

Self Assessment Question (SAQ) 4: The figure represents a disc of mass M and radius R, lying in XY- plane with its centre on X-axis at a distance 'b' from the origin. Determine the moment of inertia of the disc about Y-axis if its moment of inertia about a diameter is $\frac{MR^2}{4}$.



8.6 SUMMARY

In the present unit, we have studied about equations of motion and derived all the three equations of motion. In the unit, we have also studied Newton's laws of rotational motion. According to the first law of rotational motion "unless an external torque is applied to it, the state of rest or uniform rotational motion of a body about its fixed axis of rotation remains unaltered" while the second law states "the rate of change of angular momentum (or the rate of change of rotation) of a body about a fixed axis of rotation is directly proportional to the torque applied and takes place in the direction of the torque". According to Newton's third law of rotational motion "when a torque is applied by one body on another, an equal and opposite torque is applied by the latter on the former about the same axis of rotation". Sometimes it is difficult to calculate the moments of inertia of some specific bodies. In this unit, we have also studied and derived the general theorems on moment of inertia. These theorems are known as theorem of parallel axes and theorem of perpendicular axes. If 'I_{cm}' be the moment of inertia of a body about a parallel axis through its centre of mass, 'M' be the mass of the body and 'r' be the perpendicular distance between two axes, then moment of inertia of the body $I = I_{cm} + Mr^2$. This is the theorem of parallel axes. If I_x and I_y be the moments of inertia of a plane lamina about two mutually perpendicular axes OX and OY in the plane of the lamina and I_z be its moment of inertia about an axis OZ, passing through the point of intersection O and perpendicular to the plane of the lamina, then $I_z = I_x + I_y$. This is the theorem of perpendicular axes. These theorems make easy to find out the moments of inertia of those specific bodies. We have included examples and self assessment questions (SAQs) to check your progress.

8.7 GLOSSARY

Velocity- a vector physical quantity whose magnitude gives speed

Acceleration- increase of velocity

Unless- except, if not

External- exterior, outer

Rotational- the action of moving in a circle

Unaltered- unchanged, unaffected

Facilitate- make easy, smooth the progress of

8.8 TERMINAL QUESTIONS

1. Explain the equations of motion.

2. What are Newton's laws of rotational motion? Explain.

3. Discuss and derive general theorems on moment of inertia.

4. Calculate the moment of inertia of mass M and length L about an axis perpendicular to the length of the rod and passing through a point equidistant from its midpoint and one end.

5. Give the statement of theorem of parallel axes. Also derive this theorem.

6. State and establish the theorem of perpendicular axes.

8.9 ANSWERS

Self Assessment Questions (SAQs):

1. Since the car is moving with constant speed, therefore acceleration of car a = 0.

Here, u = 30 Km/hr = 25/3 m/sec, t = 1 hr = 3600 sec

Using second equation of motion, $s = ut + \frac{1}{2} a t^2$

$s = (25/3) \times 3600 + \frac{1}{2} (0) \times (3600)^2 = 3 \times 10^4$ m = 30 Km

2. Given $u = 6 \times 10^6$ m/sec, $a = - 1.26 \times 10^{14}$ m/sec^2, v = 0

Using third equation of motion, $v^2 = u^2 + 2as$

$(0)^2 = (6 \times 10^6)^2 + 2 (- 1.26 \times 10^{14})\, s$

or s = 0.143 m

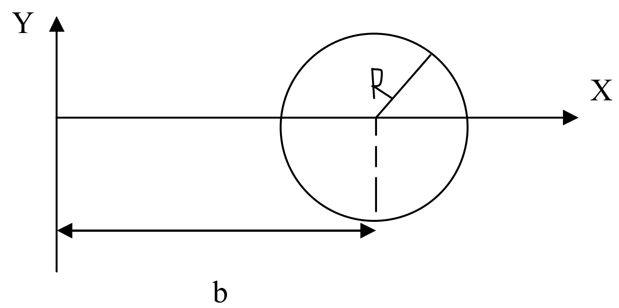
3. Here acceleration = at^2

Using v = u + at

$$v = u + (at^2)\, t = u + at^3$$

Hence relation (c) is valid.

4.



By the theorem of parallel axes, the moment of inertia of disc about Y-axis is-

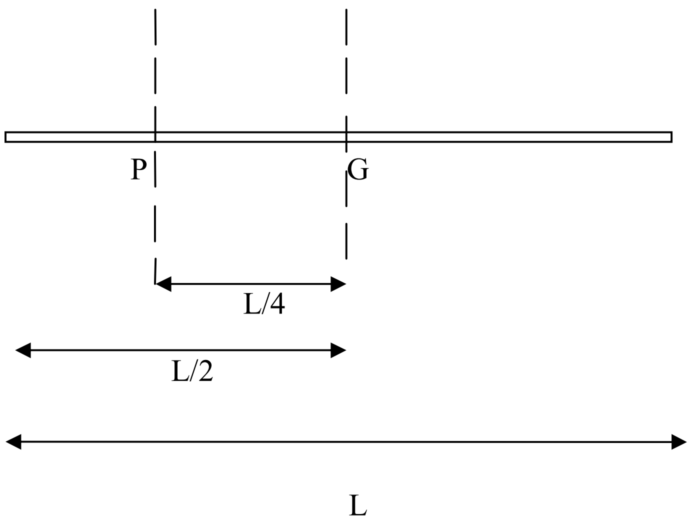
$I = I_{cm} + Mb^2$

$= \frac{MR^2}{4} + Mb^2$

$= M\left(\frac{R^2}{4} + b^2\right)$

Terminal Questions:

4. The moment of inertia of rod about an axis passing through centre of mass and perpendicular to length $I_{cm} = \frac{ML^2}{12}$



The moment of inertia of rod about an axis passing through P and perpendicular to length by theorem of parallel axes is-

$I = I_{cm} + Mr^2$

Here $r = PG = L/4$

Therefore, $I = (ML^2/12) + M(L/4)^2 = (7/48)\,ML^2$

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UNIT 9: FORMULATION OF MOMENT OF INERTIA

Structure

9.1 INTRODUCTION

In the previous unit, we have studied theorem of parallel axes and theorem of perpendicular axes. These theorems make easy the calculations of moment of inertia in some typical cases. In general, the moment of inertia of a body is calculated as the sum $\Sigma\ mr^2$ for all particles making up the body by integration or other means. In this unit, we shall formulate and derive the moment of inertia for some simple symmetric systems like rod, rectangular lamina, circular lamina, solid sphere and cylinder.

9.2 OBJECTIVES

After studying this unit, you should be able to-

- Understand the formulation and derivation of moment of inertia
- Solve problems based on moment of inertia
- Apply the formulae of moment of inertia

9.3 FORMULATION AND DERIVATION OF MOMENT OF INERTIA

The moment of inertia of a continuous homogeneous body of a definite geometrical shape can be calculated by (i) first obtaining an expression for the moment of inertia of an infinitesimal element of the same shape of mass dm about the given axis i.e. $dm.r^2$, where r is the distance of the infinitesimal element from the axis and then (ii) integrating this expression over appropriate limits so as to cover the whole body. In fact sometimes the theorem of parallel and perpendicular axes are also used to calculate the moment of inertia about an axis when the moment of inertia of the body about some other axis has first been calculated. Thus,

$$I = \int dm.r^2$$

where the integral is taken over the whole body.

9.3.1 Moment of Inertia of a Thin Uniform Rod

(i) About an axis passing through its centre of mass and perpendicular to its length

Let PQ be a thin uniform rod of mass per unit length m. Let RS be the axis passing through the centre of mass C of the rod and perpendicular to its length PQ.

Let us consider an element of length dr at a distance r from centre of mass C.

The mass of the element, $dm = m.dr$

The moment of inertia of element about the axis through C $= dm.r^2$

$= m\,dr\,r^2$

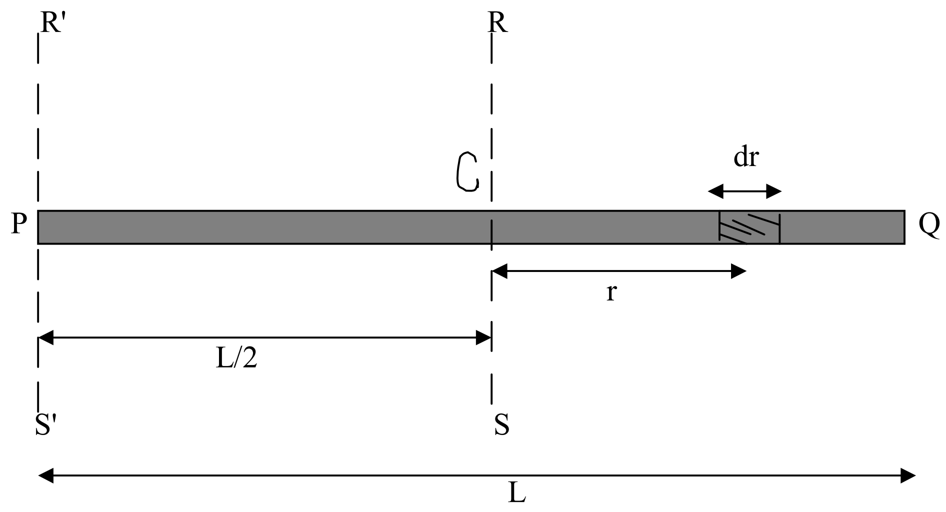

Figure 1

The moment of inertia of the whole rod about axis RS is the sum of the moments of inertia of all such elements lying between r = - L/2 at P and r = L/2 at Q. Hence the moment of inertia

$I_{cm} = \int_{-L/2}^{L/2} mr^2 dr$

$= 2m \int_0^{L/2} r^2 dr = \frac{mL^3}{12} = \frac{(mL)L^2}{12}$

$= \frac{ML^2}{12}$

where mL = M, the total mass of the rod

Thus, $I_{cm} = \frac{ML^2}{12}$ …..(1)

(ii) About an axis passing through its one end and perpendicular to its length

Let $R'\,S'$ be the axis passing through the end P of the rod (Figure 1). The moment of inertia I about a parallel $R'\,S'$ axis passing through one end (using theorem of parallel axes)-

$I = I_{cm} + M\,(CP)^2$

$= \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2$

$= \frac{ML^2}{3}$

9.3.2 Moment of Inertia of a Rectangular Lamina

(i) About an axis in its own plane, parallel to one of the sides and passing through the centre of mass

Let PQRS be a rectangular lamina of mass M, length l and breadth b with O as its centre of mass. Let the mass per unit area of the lamina is σ. Let us consider a strip of width dr parallel to the given axis Y' at a distance r from it.

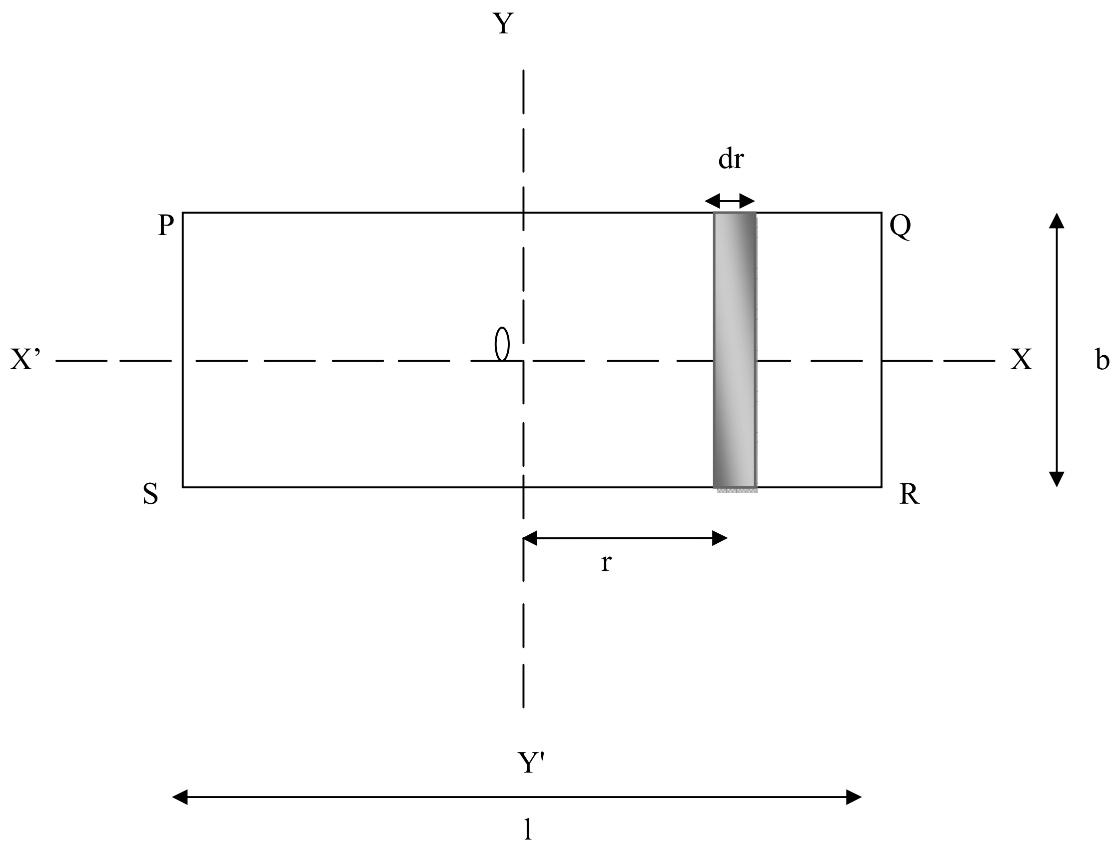

Figure 2

Area of the strip = b dr

Mass of the strip m = (bdr) σ

Moment of inertia of strip about the axis YY' = mr^2

$= (bdr)\,\sigma\, r^2$

$= \sigma\, b\, r^2\, dr$

The moment of inertia of the whole lamina about axis YY′

$I_y = \int_{-l/2}^{l/2} \sigma b r^2 dr = 2\int_0^{l/2} \sigma b r^2 dr$

$= 2\sigma b\int_0^{l/2} r^2 dr = \frac{\sigma b l^3}{12} = \frac{(\sigma b l)l^2}{12}$

i.e. $I_y = \frac{Ml^2}{12}$ …..(1)

where σbl = M, the total mass of the lamina

Similarly, the moment of inertia of the lamina about an axis XX' parallel to the side of length l and passing through the centre will be-

$I_x = \frac{Mb^2}{12}$ …..(2)

(ii) About an axis perpendicular to its plane and passing through the centre of mass

The moment of inertia of the lamina about an axis passing through the centre C and perpendicular to the plane of the lamina

$I = I_x + I_y$ (using theorem of perpendicular axes)

$= \frac{Mb^2}{12} + \frac{Ml^2}{12}$

$= M\left(\frac{l^2+b^2}{12}\right)$ …..(3)

9.3.3 Moment of Inertia of a Circular Lamina

(i) About an axis passing through its centre and perpendicular to its plane

Let O be the centre of the circular lamina and σ the mass per unit area. Let the lamina be supposed to be composed of a number of thin circular strips. Let us consider one such strip of radius r and thickness dr.

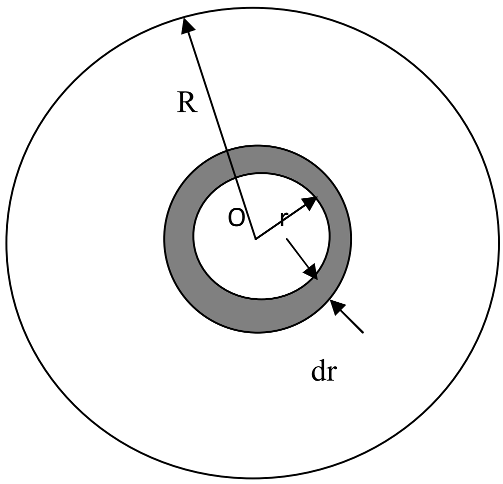

Figure 3

The circumference of the strip = 2π r

Area of the strip = 2π r dr

Mass of the strip m = σ (2π r dr)

The moment of inertia of the strip about an axis passing through O and perpendicular to the plane of the lamina dI = mass × (distance)^2 = m × r^2

$= \sigma(2\pi\, r\, dr)\, r^2$

$= \sigma\,(2\pi\, r^3\, dr)$

The moment of inertia of whole lamina I = ∫dI

$= \int \sigma\,(2\pi\, r^3\, dr)$

or $I = 2\pi\sigma \int r^3\, dr$ …..(1)

If circular lamina is solid, then its moment of inertia $I = 2\pi\sigma\int_0^R r^3 dr$

$= 2\pi\sigma \frac{R^4}{4} = \frac{1}{2}R^2(\pi R^2\sigma)$

$= \frac{1}{2}R^2 M$

where $\pi R^2\sigma = M$, mass of the disc

Thus, $I = \frac{1}{2} M R^2$ …..(2)

If lamina is having concentric hole and R' and R be the its internal and external radii then the moment of inertia $I = 2\pi\sigma\int_{R'}^{R} r^3 dr$

$= \frac{2\pi\sigma}{4}[R^4 - R'^4]$

$= \frac{2\pi\sigma}{4}(R^2 + R'^2)(R^2 - R'^2)$

$= \pi(R^2 - R'^2)\,\sigma\left\{\frac{1}{2}(R^2 + R'^2)\right\}$

$= \frac{1}{2} M (R^2 + R'^2)$

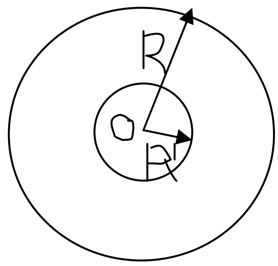

Figure 4

where $\pi(R^2 - R'^2)\,\sigma = M$, mass of the lamina

Thus, $I = \frac{1}{2} M (R^2 + R'^2)$ …..(3)

(ii) About any diameter

Let us consider two mutually perpendicular diameters AB and CD of the lamina. The lamina is symmetrical about both diameters AB and CD.

In accordance with the theorem of perpendicular axis, the moment of inertia about diameter $I_D = \frac{I}{2}$, where I is the moment of inertia of the disc, about an axis. Through it's center and perpendicular to it's plane.

For solid lamina, $I_D = \frac{1}{2}\left(\frac{1}{2}MR^2\right)$

$= \frac{1}{4}MR^2$

For circular lamina with concentric hole,

$I_D = \frac{1}{4}M(R^2 + R'^2)$

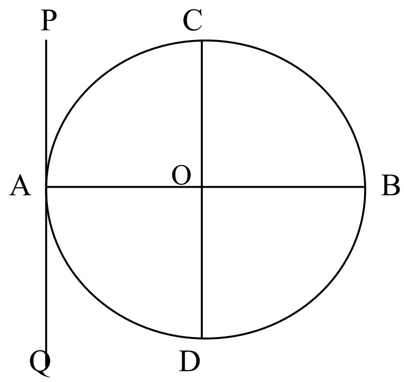

Figure 5

(iii) About a tangent in its plane

Let PQ be a tangent to the lamina in its plane and let it be parallel to the diameter CD.

Using theorem of parallel axes, the moment of inertia of lamina about PQ = Moment of inertia of lamina about CD + $(OA)^2$

or $I_T = I_D + MR^2$. M.

For a solid lamina, $I_T = \frac{1}{4}MR^2 + MR^2$

$= \frac{5}{4}MR^2$

For a lamina having a concentric hole,

$I_T = \frac{1}{4}\ M(R^2 + R'^2) + MR^2$

$= \frac{1}{4}M\ (5R^2 + R'^2)$

(iv) About a tangent perpendicular to its plane

Using theorem of parallel axes, moment of inertia of lamina about a tangent perpendicular to its plane $I'_T = I + MR^2$

For a solid lamina, $I'_T = \frac{1}{2}M\ R^2 + MR^2 = \frac{3}{2}MR^2$

For a circular lamina having concentric hole, $I'_T = \frac{1}{2}M(R^2 + R'^2) + MR^2$

$= = \frac{1}{2}M(3R^2 + R'^2)$

9.3.4 Moment of Inertia of a Solid Sphere

(i) About a diameter

Let us consider a sphere of radius R and mass M with centre at O. Let ρ be the density of the material of the sphere.

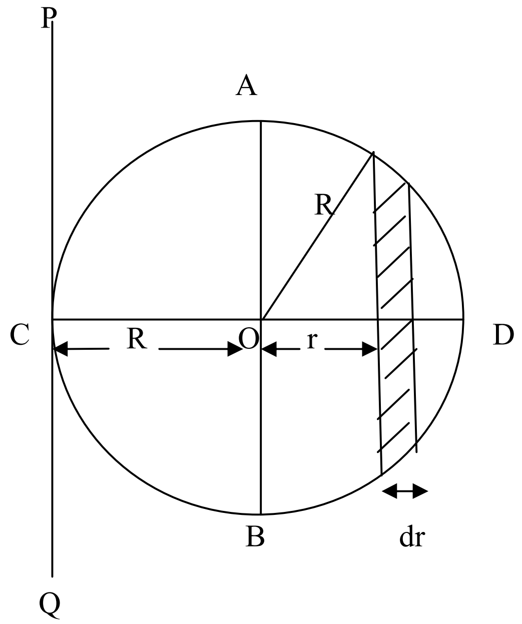

Figure 7

Let us divide the sphere into a number of thin discs by planes perpendicular to the diameter CD and let us consider one such elementary disc of thickness dr at a distance r from O.

Obviously, the radius of the elementary disc = $\sqrt{R^2 - r^2}$

Volume of the elementary disc = its area × its thickness

$= \pi(R^2-r^2)\ dr$

Mass of the elementary disc m = volume × density

$$= \pi(R^2-r^2)\ dr\ \rho$$

The moment of inertia of the disc about diameter CD-

$dI = \frac{1}{2}$ [mass of the disc × (radius of the disc)^2]

$= \frac{1}{2}\pi(R^2-r^2)\ dr\ \rho\times (R^2-r^2)$

$= \frac{1}{2}\pi(R^2-r^2)^2\ dr\ \rho$ .....(1)

The moment of inertia of the whole sphere about CD, $I = \int_{-R}^{+R} dI$

$= \int_{-R}^{+R} \frac{1}{2}\pi(R^2 - r^2)^2\rho\ dr$

$= \int_{0}^{R} \pi(R^2 - r^2)^2\rho\ dr$

$= \pi\rho\int_{0}^{R} (R^4 + r^4 - 2R^2r^2)dr$

$= \frac{8}{15}\pi\rho\ R^5$

$= \frac{2}{5}\left(\frac{4}{3}\pi R^3\rho\right)R^2$

$= \frac{2}{5}\ M\ R^2$, where $\frac{4}{3}\pi R^3\rho = M$, the mass of the sphere

Therefore, $I = \frac{2}{5}\ M\ R^2$

(ii) About a tangent

Let PQ be a tangent to the sphere parallel to the diameter AB and at a distance R (the radius of the sphere) from it.

Using theorem of parallel axes, the moment of inertia of the solid sphere about a tangent PQ-

I_T = moment of inertia of sphere about CD + $M\times(OC)^2$

$= \frac{2}{5}\ M\ R^2 + MR^2 = \frac{7}{5}\ MR^2$

9.3.5 Moment of Inertia of a Solid Cylinder

Let us consider a solid cylinder of mass M, radius R and length L.

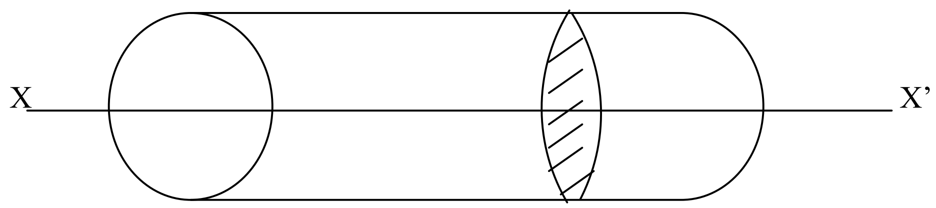

Figure 8

Volume of the cylinder = $\pi R^2 L$

Density of the cylinder $\rho = \frac{M}{\pi R^2 L}$

or $M = (\pi R^2 L)\rho$

(i) Moment of inertia of solid cylinder about geometrical axis

Let us suppose that the cylinder be formed of a large number of co-axial discs of equal radius R, the axis being the geometrical axis (XX') of the cylinder. Let us consider one such disc of mass m and radius R.

Moment of inertia of the disc about geometrical axis XX' = $\frac{1}{2}$ m R^2

Therefore, the moment of inertia of the whole cylinder about its geometrical axis XX',

$$I_1 = \Sigma \left(\frac{1}{2} mR^2\right)$$

$$= \frac{1}{2} R^2 \Sigma m$$

$$= \frac{1}{2} R^2 M$$

where $\Sigma m = M$, the mass of the cylinder

Therefore, $I_1 = \frac{1}{2} MR^2$ …..(1)

(ii) Moment of inertia about an axis passing through centre and perpendicular to geometrical axis

Let us consider an axis YY' perpendicular to geometrical axis XX' and passing through centre O of the cylinder. Let us suppose that the cylinder to be formed of coaxial discs of equal radius R.

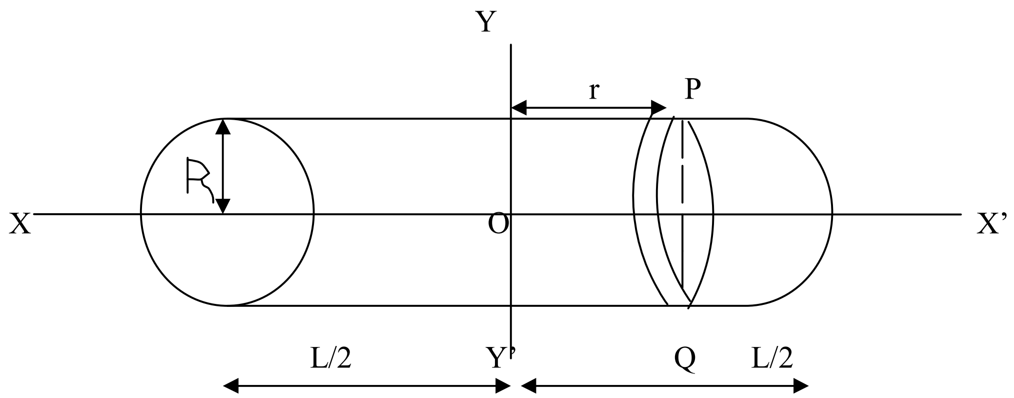

Figure 9

Let us consider one such disc of radius R and thickness dr at a distance r from the axis.

Area of disc = πR^2

Volume of the disc = (πR^2) dr

Mass of the disc $m = (\pi R^2)\, dr\, \rho$

Moment of inertia of disc about its diameter PQ = $\frac{1}{4}$ [mass of the disc × (radius of the disc)^2]

$$= \frac{1}{4}(\pi R^2\, dr\, \rho)\, R^2$$

$$= \frac{1}{4}(\pi R^4\, \rho\, dr)$$

Using theorem of parallel axes, the moment of inertia of the disc about axis YY' –

dI_2 = (Moment of inertia of disc about its diameter PQ) + $m r^2$

$$= \frac{1}{4}\pi R^4 \rho\, dr + (\pi R^2\, dr)\, \rho\, r^2$$

$$= \pi R^2 \rho \left(\frac{R^4}{4} + r^2\right) dr$$

The moment of inertia of whole cylinder about axis YY'-

$$I_2 = \int_{-L/2}^{+L/2} dI_2$$

$$= \int_{-L/2}^{+L/2} \pi R^2 \rho \left(\frac{R^4}{4} + r^2\right) dr$$

$$= 2\, \pi R^2 \rho \int_0^{L/2} \left(\frac{R^4}{4} + r^2\right) dr$$

$$= (\pi R^2 L \rho)\left[\frac{R^2}{4} + \frac{L^2}{12}\right]$$

$$= M\left[\frac{R^2}{4} + \frac{L^2}{12}\right]$$

where $\pi R^2 L \rho = M$, the mass of cylinder

Therefore, $I_2 = M\left[\frac{R^2}{4} + \frac{L^2}{12}\right]$

Example 1: The mass and radius of a solid circular disc are 500 Kg and 1 metre respectively. Calculate its moment of inertia about its axis.

Solution: The mass of circular disc M = 500 Kg

The radius of circular disc R = 1 m

Moment of inertia of the disc about its axis I = $\frac{1}{2}$ MR^2

= $\frac{1}{2}$ (500) $(1)^2$ = 250 Kg-m^2

Example 2: Determine the moment of inertia of a rectangular lamina of 2 Kg about an axis perpendicular to its plane and passing through the centre of mass. The length and breadth of lamina are 100 cm and 50 cm respectively.

Solution: Mass of lamina M = 2 Kg, l = 100 cm = 1 m, b = 50 cm = 0.5 m

Moment of inertia of lamina about an axis perpendicular to its plane and passing through the centre of mass I = $M\left(\frac{l^2+b^2}{12}\right)$

$$= 2\left[\frac{1^2+0.5^2}{12}\right] = 0.21 \text{ Kg-m}^2$$

Self Assessment Question (SAQ) 1: The moment of inertia of a disc about a tangent perpendicular to the plane of disc is-

(a) $\frac{3}{2}$ M R^2 (b) $\frac{5}{4}$ M R^2 (c) M R^2 (d) $\frac{1}{2}$ M R^2

Self Assessment Question (SAQ) 2: The moment of inertia of a thin rod of mass M and length L about an axis passing through one end and perpendicular to length is-

(a) ML^2 (b) $\frac{ML^2}{12}$ (c) $\frac{ML^2}{4}$ (d) $\frac{ML^2}{3}$

9.4 SUMMARY

In the present unit, we have formulated and derived the expressions for moment of inertia of thin uniform rod, rectangular lamina, circular lamina, solid sphere and solid cylinder. Moment of inertia of a thin uniform rod about an axis passing through its centre of mass and perpendicular to its length is expressed as $\frac{ML^2}{12}$, where M is the mass of the rod and L its length while the moment of inertia of the rod about an axis passing through its one end perpendicular to its length is derived as $\frac{ML^2}{3}$. The moment of inertia of a rectangular lamina of mass M, length l and breadth b about an axis perpendicular to its plane and passing through the centre of mass is given as $M\left(\frac{l^2+b^2}{12}\right)$.We have established the expression for moment of inertia of a circular lamina of mass M and radius R about an axis passing through its centre and perpendicular to its plane as $\frac{1}{2}$ M R^2. We have formulated the moment of inertia of a solid sphere about a diameter as $\frac{2}{5}$ M R^2 and about a tangent as $\frac{7}{5}$ M R^2. We have also derived the formulae for moment of inertia of a solid cylinder. After studying the unit, we can solve problems based on moment of inertia. We have also included examples and self assessment questions (SAQs) in the unit to check your progress.

9.5 GLOSSARY

Continuous- unbroken

Homogeneous- uniform

Infinitesimal- microscopic, extremely small

Appropriate- suitable, proper

Uniform- consistent, homogeneous

9.6 TERMINAL QUESTIONS

1. Derive the ratio of moment of inertia of a rectangular bar about an axis passing through one of its ends and through its centre.
2. Establish the expression for moment of inertia of a rectangular lamina about an axis in its own plane parallel to one of the sides and passing through the centre of mass.
3. Calculate the moment of inertia of a solid sphere about (i) a diameter and (ii) a tangent
4. Calculate the moment of inertia of a circular lamina about an axis passing through its centre and perpendicular to its plane. The mass of the lamina is 300 gm and radius 50 cm.
5. Calculate the moment of inertia of a solid cylinder about its own axis.

9.7 ANSWERS

Self Assessment Questions (SAQs):

1. Using theorem of parallel axis, $I = \frac{1}{2} M R^2 + M R^2$

$$= \frac{3}{2} M R^2$$

Hence option (a) is correct

2. Option (d) is correct

Terminal Questions:

4. Mass of lamina M = 300 gm = 0.3 Kg, Radius R = 50 cm = 0.5 m

Required moment of inertia $I = \frac{1}{2} M R^2$

$$= \frac{1}{2} \times 0.3 \times 0.5$$

$$= 0.075 \text{ Kg-m}^2$$

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Unit: 10 PENDULUMS

Structure

10.1 INTRODUCTION

In the previous units, we have studied about acceleration due to gravity, centre of gravity, radius of gyration and moment of inertia of various bodies. We have also studied the physical significance and importance of radius of gyration, moment of inertia, centre of gravity and acceleration due to gravity. In this unit, we shall also analyze the involvement of these in harmonic oscillators. In the present unit, we shall study compound pendulum and Kater's pendulum and their applications. We shall establish the expressions for time periods of these pendulums.

10.2 OBJECTIVES

After studying this unit, you should be able to-

- Understand the theory and principles of compound pendulum and Kater's pendulum
- Solve problems based on compound and Kater's pendulum
- Apply the formulae of time periods of compound and Kater's pendulum
- Understand the applications of compound pendulum and Kater's pendulum

10.3 HARMONIC OSCILLATORS

A system executing harmonic motion may be referred as harmonic oscillator. In harmonic oscillator, the frequency of the motion is not affected by the amplitude of the oscillation and it may be considered that the effect of several driving forces on the motion is linear. Compound pendulum and Kater's pendulum are the examples of the harmonic oscillators. Now, let us discuss these pendulums.

10.3.1 Compound Pendulum

Compound pendulum is also referred as physical pendulum. A compound pendulum is just a rigid body capable of oscillating in a vertical plane about a horizontal axis passing through the body but not through its centre of gravity. The point of intersection of the horizontal axis of rotation and the vertical plane through centre of gravity of the pendulum is known as 'centre of suspension'.

Let us consider a compound pendulum in the form of a rigid body of mass m suspended about a horizontal axis through the centre of suspension S. Let the distance between centre of gravity G and the point of suspension be l. In equilibrium position, the centre of gravity G is vertically below S. If we displace the body slightly to one side through an angle θ, its centre of gravity also shifts to new position G' (say) as shown in figure 1.

The weight mg of the body acts at centre of gravity G vertically downward.

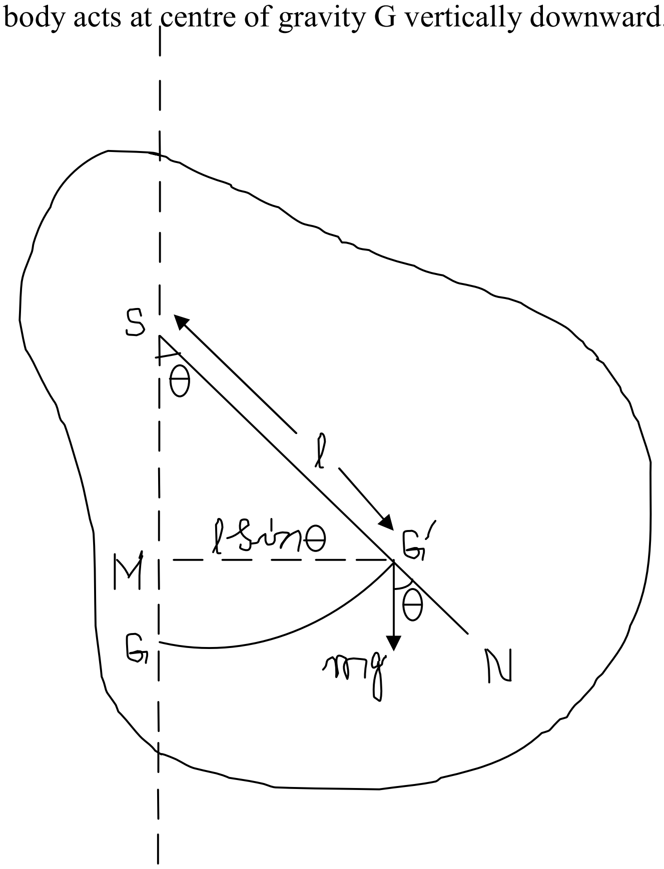

Figure 1

The restoring couple produced = - mg (MG')

$$= - mg\ (l \sin\theta) \qquad .....(1)$$

Let I be the moment of inertia of the body about the horizontal axis passing through S, then deflecting couple $\tau = I\ \alpha$

$$= I \frac{d^2\theta}{dt^2} \qquad .....(2)$$

where $\alpha = \frac{d^2\theta}{dt^2}$ is the angular acceleration of pendulum

From equation (1) and (2), we have equation of motion of compound pendulum as-

$$I \frac{d^2\theta}{dt^2} = - mg\ l \sin\theta \qquad .....(3)$$

If angular displacement θ is small, then $\sin\theta \approx \theta$, then equation (3) becomes-

$I \frac{d^2\theta}{dt^2} = - mg\, l\, \theta$

i.e. $\frac{d^2\theta}{dt^2} = - mg\, l\, \theta \;/ I$

or $\frac{d^2\theta}{dt^2} = - \omega^2\theta$ .....(3)

where $mgl/ I = \omega^2$

Therefore $\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$ .....(4)

The above equation (4) represents the differential equation of compound pendulum.

Let the solution of equation (4) be-

$\theta = A \sin (\omega_0 t + \emptyset)$ .....(5)

where A and ∅ are to be determined by initial conditions.

Differentiating equation (5), we get-

$\frac{d\theta}{dt} = A\, \omega_0 \cos (\omega_0\, t + \emptyset)$ .....(6)

Initially at t = 0, $\theta = \theta_0$ and $\frac{d\theta}{dt} = 0$

From equation (6), we get-

$0 = A\, \omega_0 \cos (\omega_0 \times 0 + \emptyset)$

or $\cos\emptyset = 0$

or $\emptyset = \frac{\pi}{2}$

Therefore equation (5) becomes-

$\theta = A \sin(\omega_0 t + \frac{\pi}{2})$

or $\theta = A \cos \omega_0 t$ .....(7)

At t = 0, $\theta = \theta_0$, therefore equation (7) gives-

$\theta_0 = A \cos \omega_0 \times 0$

or $A = \theta_0$

Thus the equation (5) becomes-

$\theta = \theta_0 \; \sin(\omega_0 t + \emptyset)$ …..(8)

which is the general solution of equation (4). Here $\emptyset$ is initial phase and θ_0 is angular amplitude.

Time period of motion $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{mgl}{I}}}$

or $T = 2\pi\sqrt{\frac{I}{mgl}}$ …..(9)

This is the expression for the period of a compound pendulum.

Let I_0 be the moment of inertia of the body about an axis passing through centre of gravity, then by the theorem of parallel axes, we get-

$I = I_0 + ml^2$

$= mk^2 + ml^2$

where k is the radius of gyration of the body about the parallel axis, through centre of gravity G.

Now the equation(9) may be expressed as-

$$T = 2\pi\sqrt{\frac{mk^2+ml^2}{mgl}}$$

$$= 2\pi\sqrt{\frac{m(k^2+l^2)}{mgl}}$$

$$= 2\pi\sqrt{\frac{l(\frac{k^2}{l}+l)}{gl}}$$

or $T = 2\pi\sqrt{\frac{(l+\frac{k^2}{l})}{g}}$ …..(10)

The time period of simple pendulum of effective length L is-

$T = 2\pi\sqrt{\frac{L}{g}}$ …..(11)

Let us compare equations (10) and (11), obviously the period of oscillations of a compound pendulum is same as that of a simple pendulum of length $L = l + \frac{k^2}{l}$. The length $L = l + \frac{k^2}{l}$ is called the length of equivalent simple pendulum.

Let us extend the line SG' upto N in such a way that-

$SN = l + \frac{k^2}{l}$

Then the point N is known as the centre of oscillation. Obviously, $G'N = \frac{k^2}{l} = l'$ (say)

Then, time period $T = 2\pi\sqrt{\frac{l+l'}{g}} = 2\pi\sqrt{\frac{L}{g}}$ …..(13)

Using compound pendulum, we can study the variation of time period T with length l and determine the value of the acceleration due to gravity (g), the position of centre of gravity and radius of gyration of the pendulum about an axis passing through centre of gravity and perpendicular to its length.

10.3.2 Kater's Pendulum

Kater's pendulum is the improved form of compound pendulum in the form of a long rod, having two knife edges K_1 and K_2 fixed near the ends facing each other but lying on opposite sides of

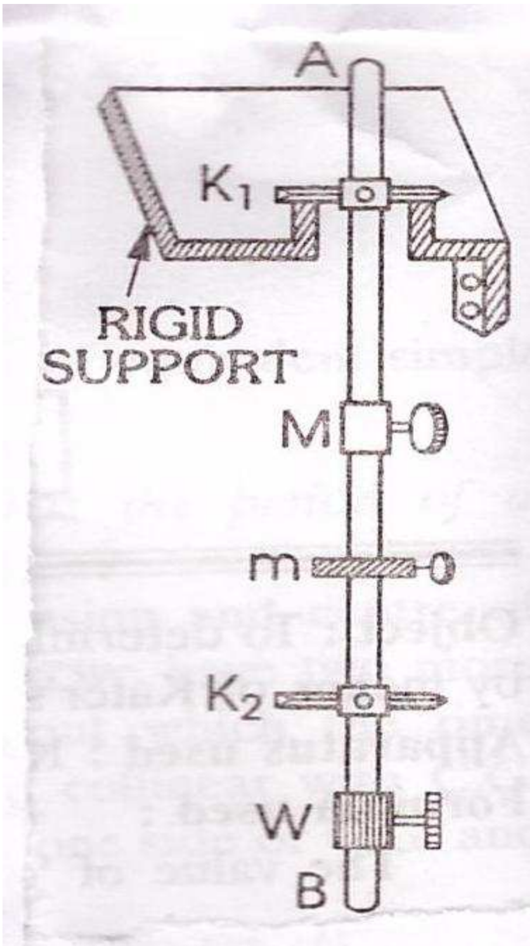

Figure 2

the centre of gravity. The position of the centre of gravity of the bar can be altered by shifting the weights M and m upward or downwards, which can be slided and fixed at any point. The smaller weight m, having a micrometer screw arrangement is used for the finer adjustment of the final position of centre of gravity. The centre of gravity lies un-symmetrically, between K_1 and K_2 due to the weight W, fixed at one end. Using Kater's pendulum, we can calculate the value of acceleration due to gravity with the help of following formula-

$$g = \frac{8\pi^2}{\left[\frac{T_1^2+T_2^2}{l_1+l_2}\right]} \qquad \ldots..(1)$$

where T_1 and T_2 be the time period about one knife edge and nearly equal time period about the other knife edge respectively. l_1 and l_2 be the distance of one knife edge from the centre of gravity of the pendulum and that of other knife edge from the centre of gravity of the pendulum. (l_1+l_2) represents the distance between two knife edges.

Example1: A metal disc of radius 0.7 m oscillates in its own plane about an axis passing through a point on its edge. Find the length of equivalent simple pendulum.

Solution: Here $I = I_G + mR^2$

$$= \frac{1}{2} mR^2 + mR^2 = \frac{3}{2} mR^2$$

and $l = R$

Time period of compound pendulum

$$T = 2\pi\sqrt{\frac{I}{mgl}} = 2\pi\sqrt{\frac{\frac{3}{2}mR^2}{mgR}}$$

$$= 2\pi\sqrt{\frac{\frac{3}{2}R}{g}}$$

Comparing with $T = 2\pi\sqrt{\frac{L}{g}}$

Therefore, the length of equivalent simple pendulum $L = \frac{3}{2}R = \frac{3}{2} \times 0.7$

$= 1.05$ metre

Self Assessment Question (SAQ) 1: Calculate the time period of a compound pendulum of mass 1.5 Kg and length 1 m. The moment of inertia of the pendulum about the horizontal axis passing through the centre of suspension is 2 Kg-m^2.

Self Assessment Question (SAQ) 2: Choose the correct option-

When centre of suspension coincides with centre of gravity of compound pendulum, the periodic time will be-

(a) zero (b) π/2 (c) infinite (d) none of these

10.4 SUMMARY

In the present unit, we have studied the theory and principles of some specific harmonic oscillators like compound pendulum and Kater's pendulum. We have seen that a compound pendulum is just a rigid body capable of oscillating in a vertical plane about a horizontal axis passing through the body but not through its centre of gravity. The point of intersection of the horizontal axis of rotation and the vertical plane through centre of gravity of the pendulum is known as 'centre of suspension'. We have established the expression for time period of compound pendulum as $T = 2\pi\sqrt{\frac{I}{mgl}}$. We have also seen that using compound pendulum, we can study the variation of time period T with length l and determine the value of the acceleration due to gravity (g), the position of centre of gravity and radius of gyration of the pendulum about an axis passing through centre of gravity and perpendicular to its length. In the unit, we have seen that Kater's pendulum is the improved form of compound pendulum in the form of a long rod, having two knife edges fixed near the ends facing each other but lying on opposite sides of the centre of gravity. Using Kater's pendulum, we can calculate the value of acceleration due to gravity with the help of formula

$$g = \frac{8\pi^2}{\left[\frac{T_1^2+T_2^2}{l_1+l_2}\right]}$$

where T_1 and T_2 be the time period about one knife edge and nearly equal time period about the other knife edge respectively. l_1 and l_2 be the distance of one knife edge from the centre of gravity of the pendulum and that of other knife edge from the centre of gravity of the pendulum. (l_1+l_2) represents the distance between two knife edges. We have included examples and self assessment questions (SAQs) in the unit to check your progress.

10.5 GLOSSARY

Executing- carrying out a plan, performing a skilful action or manoeuvre

Referred- mentioned

Driving – lashing, motivating

Rigid- inflexible, stiff

Capable- able, competent

Adjustment- alteration, amendment

10.6 TERMINAL QUESTIONS

1. What is a harmonic oscillator? Describe the theory of compound pendulum. How can the time period be calculated by compound pendulum?

2. Describe the construction and working of Kater's pendulum. Give the necessary formula to calculate acceleration due to gravity by Kater's pendulum.

3. Calculate the time period of a compound pendulum of mass 1 Kg and length 1.2 m. The moment of inertia of the pendulum about the horizontal axis passing through the centre of suspension is 1 Kg-m^2.

10.7 ANSWERS

Self Assessment Questions (SAQs):

1. Given, m = 1.5 Kg, l = 1 m, I = 2 Kg-m^2

Using $T = 2\pi\sqrt{\frac{I}{mgl}}$

$$= 2 \times 3.14 \sqrt{\frac{2}{1.5\times 9.8\times 1}} = 2.32 \text{ sec}$$

2. (c) infinite

Terminal Questions:

3. Given m = 1 Kg, l = 1.2 m, I = 1 Kg-m^2

Using formula $T = 2\pi\sqrt{\frac{I}{mgl}}$, we get-

$$T = 2\times 3.14\sqrt{\frac{1}{1\times 9.8\times 1.2}} = 1.83 \text{ sec}$$

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UNIT 11: GRAVATION

Structure

11.1 INTRODUCTION

Since the earliest times, gravity meant the tendency of most bodies to fall to earth. In reverse, things that leaped upwards, like flames of fire, were said to have “levity”. Aristotle was the first writer to attempt a measurable description of falling motion: he wrote that an object fell at a constant speed, attained shortly after being released and heavier things fell faster in proportion to their mass. Of course this is rubbish, but, falling motion is rather fast—it’s hard to see the speed variation when you drop something to the ground. Galileo was the first to get it right. He realized that a falling body picked up speed at a constant rate— in other words, it had constant acceleration. He also made the crucial observation that, if air resistance and buoyancy can be neglected, all bodies fall with the same acceleration, bodies of different weights dropped together reach the ground at the same time. It was Sir Isaac Newton who not only provided this explanation in his famous inverse square law of gravitation, but managed to create the explanation of motion on earth and motion in the heavens. This had profound philosophical and scientific consequences. The unification into what became the laws of gravitation became a symbol of the predictive and quantitative power of science. The fact that a single law could explain the motion of a cannonball and the motion of Mars revolutionized understanding of our place in the universe.

In this first unit, we shall understand what we mean by gravity. We shall learn how to describe Newton’s law of gravitation. We will discuss the effect of gravitational acceleration at different altitude, effect of rotation of earth, shape of earth. Also, we shall study the gravitational field, gravitational potential and potential energy and their application.

11.2 OBJECTIVES

After studying this Unit, you should be able to-

- define gravitation
- apply Newton’s law of gravitation
- solve problems using Newton’s law of gravitation
- understand the concept of a field
- recall and use the relationship that describes the gravitational force between two masses
- describe the Earth’s gravitational field and explain how the field strength varies with distance from the centre of the Earth
- Solve problems on gravitational potential and potential energy.

11.3 WHAT IS GRAVITATION?

Gravitation is a natural phenomenon by which all things with energy are brought towards one another, including stars, planets, galaxies and even light and sub-atomic particles. Gravity is answerable for many of the structures in the Universe, by creating spheres of hydrogen, where hydrogen fuses under pressure to form stars and grouping them into galaxies. On Earth, gravity gives weight to physical objects and causes the tides. Gravity has an infinite range, although its effects become gradually weaker on farther objects.

Gravity is the weakest of the four fundamental interactions of nature. The gravitational attraction is approximately 10^{-38} times the strength of the strong force necessary for the binding of nuclei that is gravity is 38 orders of magnitude weaker, 10^{-36}times the strength of the electromagnetic force, and 10^{-31} times the strength of the weak force necessary for the beta decay. As a result, gravity has a negligible influence on the performance of subatomic particles, and plays no role in determining the internal properties of everyday matter mostly governed by electromagnetic forces. On the other hand, gravity is the dominant interaction at the macroscopic scale, and is the cause of the formation, shape, and trajectory of astronomical bodies. It is responsible for various phenomena observed on Earth and throughout the universe; for example, it causes the Earth and the other planets to orbit the Sun, the Moon to orbit the Earth, the formation of tides, and the formation and evolution of galaxies, stars and the Solar System.

11.4. NEWTON'S LAW OF GRAVITATION

A point mass is one that has a radial field, like that of the Earth. Although the Earth is a large object, on the scale of the Universe it can be considered to be a point mass. The gravitational field strength at its centre is zero, since attractive forces pull equally in all directions. Beyond the surface of the Earth, the gravitational force on an object decreases with increasing distance. When the distance is measured from the centre of the Earth, the size of the force follows an inverse square law, doubling the distance from the centre of the Earth decreases the force to one quarter of the original value.

Gravity is the weakest force we know, but it is the force of gravity that controls the evolution of the universe. Everybody in the universe attracts every other body. Newton proposed that the magnitude of this force is given by

$$F = G\frac{m_1 m_2}{r^2}$$

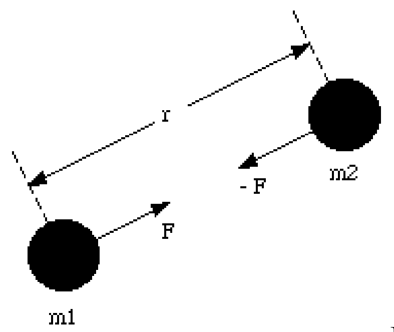

Figure 1

Where m_1 and m_2 are the masses of the particles, r is the distance between them and G is a universal constant whose value is G = 6.67 x 10^{-11} N m^2/kg^2

The gravitational forces between two particles act along the line joining them, and form an action-reaction pair.

In real life we are not dealing with point particles; instead we are dealing with extended objects. To evaluate the gravitational force between extended objects, the shell theorem can be used that is a uniform shell of matter attracts an external particle as if all the shell's mass were concentrated at its centre.

11.4.1 The Value of g Depends on Location:

Newton exactly saw this as an approval of the inverse square law. He proposed that a universal force of gravitation F existed between any two masses m and M, directed from each other, proportional to each of them and inversely proportional to the square of their separation distance **r**. In a formula, ignoring for the vector character of the force:

$$F = G\frac{Mm}{r^2}$$

$$mg = G\frac{Mm}{r^2}$$

Suppose M is the mass of the Earth, R its radius and m is the mass of some object near the Earth's surface. Then we have (replacing r by R)

$$g = G\frac{M}{R^2}$$

The capital G is known as the constant of universal gravitation**.** That is the number we need to know in order to compute the gravitational attraction between, say, two spheres of 1 kilogram each. Unlike the attraction of the Earth, which has a huge mass M, such a force is quite small, and the number G is likewise very very small. Measuring that small force in the lab is a delicate and difficult act.

11.4.2 Variation of g with Altitude

Let P be the point at an altitude *h* above the surface of the earth, let the mass of the earth be M and radius of the earth be R. Considering the earth as spherical body, the acceleration due to gravity at earth surface is given by

$$g = G\frac{M}{R^2}$$

Let the body be placed at height h from the earth surface, the acceleration due to gravity is

$$g_h = G\frac{M}{(R+h)^2}$$

Dividing these two equations, we get

$$\frac{g_h}{g} = \frac{R}{(R+h)^2}$$

Or, we get on simplification

$$g_h = g\frac{R}{(R+h)^2}$$

Which shows that the value of g decreases in height above the surface of earth.

11.4.3 Variation of g with Depth

Les us assume the earth to be a homogenous sphere having uniform density of radius R and mass M. Let ρ be the density of the earth. On the earth surface the gravity is given by

$$g = G\frac{M}{R^2}$$

Where $M = \frac{4}{3}\pi R^3 \rho$

Or, $g = G\frac{\frac{4}{3}\pi R^3 \rho}{R^2}$

Now the body is taken to a depth *h* below the earth surface. Let the acceleration due to gravity at that point be g_h

The force of gravity acting on the body is only due to inner sphere of radius (R-h)

Therefore,

$$g_h = G\frac{M^{'}}{(R-h)^2}$$

Where M' is the mass of inner solid sphere of radius (R-h).

$$M^{'} = \frac{4}{3}\pi(R-h)^3\rho$$

Hence, $g_h = G\dfrac{\frac{4}{3}\pi(R-h)^3\rho}{(R-h)^2}$

$$g_h = G\frac{4}{3}\pi(R-h)\rho$$

On dividing, we get

$$\frac{g_h}{g} = (1-\frac{h}{R})$$

Or, $g_h = g(1-\dfrac{h}{R})$

It is clear by this equation that if h increases, g_h must decrease because g is constant. Thus we conclude that the value of acceleration due to gravity decreases with increase of depth.

Special Cases:

If d=R that is at the centre of the earth

$g_h = g(1-\dfrac{h}{h}) = 0$, which explains that if the m mass of the body is at the centre of the earth, its weight will be zero.

11.4.4 Variation of g with Shape of the Earth

The earth is not a perfect sphere. It is flattened at the poles, where altitude is 90^0 and lumps at the equator where altitude is 0^0. The equatorial radius R is greater than the polar radius by nearly 21 kilometre.

We have $g = G\frac{M}{R^2}$, since G and M (mass of earth) is the constant, therefore, $g \propto \frac{1}{R^2}$, thus the value of g at a place on the surface of the earth varies inversely as the square of the radius of the earth at that place. Since the value of R is greatest at the equator, therefore the value of g is least and at the poles R is least as a result g is maximum at the poles.

11.4.5 Variation of g with Rotation of Earth

Rotation also effects the value of g. Due to the rotation of earth any particle at the surface of the earth describes a circle of radius r. the acceleration at any point on the earth surface is given by

$$g^{'} = g - r\omega^2 \cos^2\theta$$

Where ω is the angular velocity and θ is the latitude in which the point is situated. As θ increases, cos θ decreases and g will increase. So, the value of g increases as we move equator to pole. At equator, $\theta = 0$, $g^{'}=g-R\omega^2$

At poles $\theta = 90^0$ and g=g′

So, the value of acceleration due to gravity is maximum at poles. Moreover, the value of g at the poles will remain the same whether the earth is rotating or stationary.

Example 1. What is the gravitational force that the sun exerts on the earth? The earth on the sun? In what direction do these act? Where, $M_e = 5.98\times10^{24}$ kg, $M_s = 1.99\times10^{30}$kg and the earth-sun distance is 150×10^9 meters.

Solution: The force acts along the direction such that it attracts each body radially along a line towards their common center of mass. For most practical purposes, this means a line connecting the center of the sun to the center of the earth. The magnitude of both forces is the same, as we would expect from Newton's Third Law, and they act in opposite directions, both attracting each other mutually. The magnitude is given by using the law of gravitation

$$F = G\frac{m_1 m_2}{r^2}$$

$$F = G\frac{M_e M_s}{r^2}$$

On putting the values in the formula, we get 3.53 x 10^{22} N.

Example 2. Determine the mass of earth using the Newton's law of gravitation.

Solution: It is possible to measure the mass of the Earth M using the Law of Gravitation. The usual method of putting an object on a balance to determine the mass of the Earth is apparently out of the question. Start by considering a mass m on the surface of the Earth. (The numerical value of m is not important since it will cancel out in the calculation). Using the relation

$$F = G\frac{m_1 m_2}{r^2}$$

$$mg = G\frac{Mm}{R^2}$$

$$g = G\frac{M}{R^2}$$

Therefore,

$$M = \frac{gR^2}{G}$$

Where, R is the radius of the earth. On putting the values in the above equation, we get

g = 6.01811 x 10^{24} kg

Example 3: Calculate the acceleration due to gravity on the Moon's Surface.

You could take the mass m to the Moon and the mass would remain the same but the force of gravity would be less. The force of gravity would be W=mg where g is the acceleration of gravity on the Moon and also the force of gravity acting on the mass m is given by the Law of Gravitation W=F=*G m*M/R^2 , where M is the mass of the Moon.

$$mg = G\frac{Mm}{R^2}$$

$$g = G\frac{M}{R^2}$$

Putting the values of mass of moon, G and radius of the moon, we get

g = 1.62298 ms^{-2}

Which is equal to $1/6^{th}$ of the acceleration on the earth surface.

Self Assessment Question (SAQ) 1: The Value of g decreases in both cases as we go above the earth surface or when we go below the earth surface. Show that the value of gravity at height h is same as the value of acceleration due to gravity at a depth 2h.

Self Assessment Question (SAQ) 2: A 2.5 kg rock is located on the Earth's surface. If the mass of the moon is 7.4 x 10^{22} kg, and the separation distance between the centre of the rock and the centre of the moon is 3.8 x 10^{8} meters, what gravitational force does the moon exert on the rock?

11.5 APPLICATION OF LAW OF GRAVITATION

11.5.1 Free Fall Acceleration

If you throw something vertically upward and could somehow eliminate or ignore the effect of drag and air on the object,then the object accelerates constantly when it goes up and falls down,this is called free fall.If something falls freely under the effect of earth's gravity without any effect of air then the phenomenon is called free fall.While the free fall, no matter how big, small or weighty the object is, every object feel the same constant acceleration , the constant acceleration during free fall is called free fall acceleration.The free fall acceleration of earth is denoted by g and its value at the surface of earth is approximately 9.8 m/s^2,but you should also note that the value of g varies slightly with change in latitude and elevation from the surface of earth.

If the mass density of the earth depends only on the distance from the centre of the earth, considering earth as consisting of homogeneous shells, we can easily calculate the net gravitational force acting on a particle of mass m, located at an external point, a distance r from the centre of the earth. By Newton's law of gravitation

$$F = G\frac{Mm}{r^2}$$

Where, M is the mass of the earth. For a particle on the earth surface, r = R_e, the gravitational force is given by

$$mg = G\frac{Mm}{R_e^{\ 2}}$$

$$g = G\frac{M}{R_e^{\ 2}}$$

We conclude that the free-fall acceleration depends on the mass of the earth and its radius.

11.5.2 Tides

In addition to the force of gravity from the earth, every object on the earth must necessarily feel a force from the moon and the sun. However, the earth is in free fall in relation to both these bodies. Just like the astronaut on the space shuttle near the earth, the effects of the pull due to the sun and earth are cancelled out because of the free fall. Yet this cancellation is not exact, a small net force is exerted by both the moon and the sun on all objects on the earth. For objects fixed to the surface, this force is not significant. However, it does act on the oceans, causing them to lump toward the moon or sun, where the moon is closest to the earth and the force is strongest, and to lump away where the force is weaker on the opposite side from the moon. As the earth rotates on its axis, the region facing the moon changes, causing the earth to shift slightly under the oceans. This effect accounts for the daily rise and fall of the tides

***Common misconceptions about guns:**

- A dropped bullet will hit the ground before one which is fired from a gun.Gravity acts the same way on bullets, giving them the same downward acceleration and making them strike the ground at the same time if the bullet is fired horizontally over level ground.
- Bullets fired from high-powered rifles drop only a few inches in hundreds of yards.

Fired at twice the speed of sound, a bullet will drop over say 3 inches in 100 yards, and at 300 yards downrange will have dropped about 30 inches. Ammunition makers contribute to this misconception by stating the drop of their projectiles as just the extra drop caused by frictional drag compared to an ideal frictionless projectile.

11.6 GRAVITATIONAL FIELD

A field is a region of space where forces are exerted on objects with certain properties. There are three types of field

- gravitational fields affect anything that has mass
- electric fields affect anything that has charge
- magnetic fields affect permanent magnets and electric currents.

These three types of field have many similar properties and some important differences. There are key definitions and concepts that are common to all three types of field. Radiant energy such as light has mass and so is affected by gravitational fields.

Newton realised that all objects with mass attract each other. This seems surprising, since any two objects placed close together on a desktop do not immediately move together. The attractive force between them is tiny, and very much smaller than the frictional forces that oppose their motion.

Gravitational attractive forces between two objects only affect their motion when at least one of the objects is very massive. This explains why we are aware of the force that attracts us and other objects towards the Earth – the Earth is very massive. The mass of the Earth is about 6×10^{24} kg.

The strength of gravitational field at a point is defined as the force experienced by a unit mass placed at that point in the field. It, may also be defined as rate of change of gravitational potential or the potential gradient at that point. That is

$$E = -\frac{dV}{dx}$$

This shows that:

- gravitational forces are always attractive – the Earth cannot repel any object
- the Earth's gravitational pull acts towards the centre of the Earth
- the Earth's gravitational field is radial; the field lines become less concentrated with increasing distance from the Earth.
- The force exerted on an object in a gravitational field depends on its position. The less concentrated the field lines, the smaller the force. If the gravitational field strength at any point is known, then the size of the force can be calculated
- The gravitational field strength is at any point in the gravitational field is force per unit mass at that point $g=F/m$
- Close to the Earth's surface, g has the value of 9.8 N/Kg^{-1}, though the value of 10 N/kg^{-1} is used in calculations.

11.6.1 Gravitational field strength

Gravitational field strength is a vector quantity: its direction is towards the object that causes the field. Gravitational field strength is a property of any point in a field. It can be given a value whether or not a mass is placed at that point. Like gravitational force, beyond the surface of the Earth the value of $\vec{g}$ follows an inverse square law.

Because the inverse square law applies to values of g when the distance is measured from the centre of the Earth, there is little change in its value close to the Earth's surface. Even when flying in an aircraft at a height of 10 000 m, the change in distance from the centre of the Earth is minimal, so there is no noticeable change in g. The radius of the Earth is about 6.4×10^6m, so you would have to go much higher than aircraft-flying height for g to change by 1%.

The same symbol g is used to represent:

- gravitational field strength
- Free-fall acceleration.

These are not two separate quantities, but two different names for the same quantity. Gravitational field strength g is defined as the force per unit mass, g = F/m.

11.7 GRAVITATIONAL POTENTIAL

When an object changes its position relative to the Earth, there is a change in potential energy. It is not possible to place an absolute value on the potential energy of any object when h is measured relative to the surface of the Earth. Two similar objects placed at the top and bottom of a hill have different values of potential energy, but relative to the ground the potential energy is zero for both objects.

Absolute values of potential energy are measured relative to infinity. In this perspective, infinity means at a distance from the Earth where its gravitational field strength is so small as to be negligible. On an absolute scale of measurement, zero must be the smallest possible value.

The car at the top of the hill has more potential energy than the one at the bottom, but relative to ground level they both have, zero potential energy. Using this reference point

- All objects at infinity have the same amount of potential energy that is zero.
- Any object closer than infinity has a negative amount of potential energy, since it would need to acquire energy in order to reach infinity and have zero energy.

Work has to be done to move an object from within the Earth's gravitational field to infinity. Just as gravitational field strength is used to place a value on the gravitational force that would be experienced by a unit mass at any point in a gravitational field; the concept of gravitational potential is used to give a value for the potential energy. Since gravitational potential is the gravitational potential energy per unit mass placed at a point in a field, measured relative to infinity.Gravitational potential V is given by the relationship $V = U/m$, where U is the gravitational potential energy and m and M are two different masses.

Or, $V = -\frac{GMm}{r}/m$

That is $V = -\frac{GM}{r}$

So if the potential at any point in a field is known, the potential energy of a mass placed at that point can be calculated by multiplying the potential by the mass.

11.8. GRAVITATIONAL POTENTIAL ENERGY

We have discussed the relation between the force and the potential energy. Consider two particles of masses m_1 and m_2, separated by a distance r. In the gravitational field it is convenient to define the zero potential energy configuration to be one in which the two particles are separated by a large distance (infinity). Suppose the two masses are brought together say at a distance r from infinity, along the path connecting the centres of the two masses.

The gravitational potential energy measured relative to infinity of a mass, m, placed within the gravitational field of a spherical mass M. The work done by the gravitational force can be calculated as

$$W = \int_{\infty}^{r} \vec{F}.d\vec{r}$$

$$W = -G\int_{\infty}^{r} \frac{Mm}{r^2} dr$$

Or, $U = W = -G\dfrac{Mm}{r}$

$$U = -\frac{GMm}{r}$$

Where, r is the distance between the centres of mass and G is the universal gravitational constant. Gravitational potential energy is measured in joules.

The total potential energy of a system of particles is sometimes called the binding energy of the system. The total potential energy is the amount of work that needs to be done to separate the individual parts of the system and bring them to infinity.

The potential energy is always negative and is a property of the two masses together rather than of either mass alone.

The work done by the gravitational force depends only on its initial and its final position, and not on the actual path followed. For example, a baseball travels from point A to point B. The work done by the gravitational force on the baseball along the arcs is zero since the force and displacement are perpendicular. The only segments that contribute to the work done are those segments along the radial direction. The work done is negative if the force and the displacement are pointing in the opposite direction; if the force and the displacement are pointing in the same direction the work is positive therefore the net work done if we travel along the radial direction back-and-forth is zero. We can now easily show that the net work done by the gravitational force on the baseball is just determined by its initial radial position and its final radial position.

If the system contains more than two particles, the principle of superposition applies. In this case we consider each pair and the total potential energy is equal to the sum of the potential energies of each pair.

Example 4: The mass of a typical astronaut plus spacesuit is 80 kg. What would be the gravitational force acting on such an astronaut standing on the surface of Mars? State whether an astronaut on Mars would feel lighter or heavier than on Earth.

Solution: $F = GMm/r^2$

$=(6.67 \times 10^{11}\ N\ m^2\ kg^{-2}) \times 80\ kg \times (6.42 \times 10^{23}\ kg) / (3.38 \times 10^6\ m)^2 = 300\ N$

Astronaut would feel lighter.

Example 5: What is the gravitational potential energy of the moon with respect to the earth? The mass of the moon is 7.35×1022kilograms and the mass of the earth is 5.98×1024 kilograms. The earth moon distance is 384 400 kilometres.

Solution: $U = -\frac{GMm}{r}$

On putting the given values in the formula, we get = -7.63×10^{28} Jules

Example 6: What is the gravitational potential with respect to the sun at the position of the earth? The mass of the sun is 1.99×1030kilograms and the mass of the earth is 5.98×1024 kilograms. The mean earth-sun distance is 150×106 kilometers.

Solution: $V = \frac{GM}{r}$

On putting the values we get, $V=8.85 \times 10^8$ J/kg

Self Assessment Question (SAQ) 3: What is the total energy of a 90 kilogram satellite with a perigee distance 595 kilometres and apogee distance 752 kilometers, above the surface of the earth? The mass of the earth is5.98×1024 kilograms and its radius is 6.38×106 m.

Self Assessment Question (SAQ) 4: Calculate the orbital energy and orbital speed of a rocket of mass 4.0×103 kilograms and radius 7.6×103 kilometers above the center of the earth. Assume the orbit is circular. (M e = 5.98×1024kilograms).

Self Assessment Question (SAQ) 5:A satellite of mass 1000 kilograms is launched with a speed of 10 km/sec. It settles into a circular orbit of radius 8.68×103km above the center of the earth. What is its speed in this orbit? (M e= 5.98×1024 and r e = 6.38×106 m).

Self Assessment Question (SAQ) 6: What is the force of gravity acting on a 2000 kg spacecraft when it orbits two earth radii from the Earth's centre (that is the distance r_e = 6380 km above the earth surface, mass of the earth M_e= 5.98 x 10^{24} kg?

Self Assessment Question (SAQ) 7: Find the net force on the moon due to the gravitational attraction of both the earth and the Sun, assuming they are at right angles to each other (m_m= 7.35 x 10^{22} kg, m_e= 5.98 x 10^{24} kg, m_s= 1.99 x 10^{30} kg).

Self Assessment Question (SAQ) 8: calculate the effective value of g on the top of Mt. Everest, 8850 m above the sea level. That is, what is the value to gravity of objects allowed to fall freely at this height?

Self Assessment Question (SAQ) 9:

11.9 EQUIPOTENTIAL SURFACE

A surface, at all point of which the gravitational potential is same is called an equipotential surface. If, we visualise a hollow sphere of radius r with a body of mass m at the centre, the potential at each point on it will be the same. The surface of the sphere is thus an equipotential surface. Because, the difference of potential between any two points on the equipotential surface is zero, no work is done against the gravitational force in moving a unit mass along it. Or, we can say that there is no component of the gravitational field along an equipotential surface. It means that the field is at every point perpendicular to it.

The potential energy of a satellite in a circular orbit around the Earth remains constant provided that its distance from the centre of the Earth does not change. To move to a higher or lower orbit the satellite must gain or lose energy. The satellite travels along an equipotential surface, the spherical shape consisting of all points at the same potential.

For a satellite in an elliptical orbit, there is an interchange between kinetic and potential energy as it travels around the Earth.

- Equipotential surfaces around a spherical mass are also spherical.
- The spacing distance between the equipotential surfaces increases with increasing distance from the centre of earth.

11.10 SUMMERY

The Universal Law of Gravitation has several important features. First, it is an inverse square law, meaning that the strength of the force between two massive objects decreases in proportion to the square of the distance between them as they move farther apart. Second, the direction in which the force acts is always along the line connecting the two gravitating objects. Moreover, because there is no negative mass, gravity is always an attractive force. It is also noteworthy that gravity is a relatively weak force. Among the four fundamental forces in nature, the Strong and

Weak Nuclear forces, the Electromagnetic force and gravity; The gravity is the weakest. This means that gravity is only significant when very large masses are being considered. We have discussed the variation of g with altitude. How and why the acceleration due to gravity changes. Why, due to rotation of earth the value of g is affected, suppose you are at the pole of earth or at the equator of earth, the value of g are different.

We have discussed the law of Universal Gravitation together with Newton's Laws of Motion to discuss a variety of problems involving the motion of large objects like the Earth moving in orbit about the Sun.Earth's moon is held in orbit by an attractive gravitational force between Earth and the moon. Tides on Earth are due mainly to the gravitational pull of the moon on Earth. Any two masses whether or not Earth and the moon, experience a mutual gravitational force that tends to pull them together. A mass in a position to be pulled to another position by a gravitational force has gravitational potential energy. Anything water, a book, a molecule in the atmosphere, etc. has gravitational energy if it is in a position to move closer to the center of Earth. Usually, something has to do work to get the object to the elevated position. Also, many live examples are discussed to clear the gravitation, gravitational potential and gravitational potential energy.

Many solved examples are given in the unit to make the concepts clear. To check your progress, self assessment questions (SAQs) are given place to place.

11.11 GLOSSARY

Location-position

Equipotential- same potential

Consequences-significances

Revolution-transformation

Trajectory-path

Extended-lengthy

Altitude-height

Homogenous-same

Radiant-glowing

Perspective-view

Negligible-tiny

11.12 TERMINAL QUESTIONS

(Data Required: $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}, mass of the Earth = 6.0×10^{24} kg, radius of the Earth = 6.4×10^6 m, mass of the Sun = 2.0×10^{30} kg, average distance from the Earth to the Sun = 1.5×10^{11} m.)

1. The radius of the earth is 6.4 x 10^6 m and the gravitational field strength at its surface is 10 N kg^{-1}. At what height above the surface of the earth is the gravitational field strength equal to 2.5 N kg^{-1}
2. Two 2.5 kg masses are placed with their centres 10 cm apart. Calculate the gravitational attractive force between them.
3. The mass of moon is 7.4 x 10^{22} kg and its radius is 1.7 x 10^6 m. calculate the value of free fall acceleration at the moon's surface.
4. Communications satellites orbit the Earth at a height of 36 000 km. How far is this from the centre of the Earth? If such a satellite has a mass of 250 kg, what is the force of attraction on it from the Earth?
5. What is the force of attraction between the Earth and the Sun? Mass of the Sun = 2 x 1030 kg, mass of the Earth = 6 x 1024 kg, distance from the Earth to the Sun = 1.5 x 1011 m
6. The average force of attraction on the Moon from the Sun is 4.4×10^{20} N. Taking the distance from the Sun to the Moon to be about the same as that from the Sun to the Earth, what value of mass does this give for the Moon?
7. What is the force of attraction between two people, one of mass 80 kg and the other 100 kg if they are 0.5m apart?
8. According to Coulomb's Law, if the charge of either the nucleus or the orbital electron were greater, the force between the nucleus and electron would be (greater) (less).
9. According to Coulomb's Law, if the distance between the nucleus and the electron were greater the force would be (greater) (less).
10. According to Coulomb's Law, if the distance between the nucleus and electron were doubled, the force would be (1/4 as much) (1/2 as much) (two times as much) (4 times as much).
11. George, who has a mass of 75 kg, exerts gravitational force of 2.25 E -8 N on Ginger, who has a mass of 55 kg. Find the separation distance between George and Ginger.

11.13 ANSWERS

Self Assessment Questions (SAQs):

1. at height, $g_h = g\dfrac{R}{(R+h)^2}$

At Depth, $g_h = g(1 - \frac{h}{R})$

If we solve these two equations, we get when h (depth) =2h, change in gravity is same.

2. m_1 = 2.5 kg

m_2 = 7.4 E 22 kg

d = 3.8 E 8 m

F =?

F = $(Gm_1m_2)/d^2$ = [(6.67 x 10^{-11}) (N*m^2/kg^2) (2.5 kg) (7.4 E 22 kg)]/ (3.8 x 10^8 m)^2

F = 8.55 x 10^{-5} N

3. The total energy of a satellite in orbit is given by $U = -\frac{GMm}{r}$

The perigee distance from the center of the earth is 595000 + 6.38×10^6m and the apogee distance is 752000 + 6.38×10^6. The semi-major axis length is given by 595000 + 752000 + 2×6.38×10^6 = 7.05×10^6m.

U= 2.55 x 10^9 Jules

4. The total orbital energy of a circular orbit is given by: $U = -\frac{GMm}{r}$ = - 1.05×10^{11}Joules. This is also equal to T=1/2mv^2so we can find the orbital speed as $\sqrt{\frac{2T}{m}}$ = 7.2×10^4m/s.
5. It is based upon the conservation of energy. The initial kinetic energy is given by 1/2mv^2 = 1/2×1000× $(10000)^2$ = 5×$10^1$0 Joules. It also has some initial gravitational potential energy associated with its position on the surface $U = -\frac{GMm}{r}$ - 6.25×$10^1$0 Joules. The total energy is then given by $E = T$ + U = - 1.25×10^{10} Joules. In its new orbit the satellite now has a potential energy $U = -\frac{GMm}{r'}$ = - 4.6×10^{10} Joules. The kinetic energy is given by $T = E–U$ = (- 1.25 + 4.6) ×10^{10} = 3.35×10^{10} Joules. We can easily now find the velocity: $\sqrt{\frac{2T}{m}}$ = 8.1×10^3 m/s.
6. At the surface of the earth, F_g= mg. at a distance from the Earth centre of $2r_e$,
   F_g= (1/4) mg
   = (1/4) x 2000X 9.8
   = 4900 N
7. Gravitational force on the moon due to earth is

$$F_{me} = -\frac{GMm}{r^2}$$

$$= = \frac{6.67\times10^{-11}\times7.35\times10^{22}\times5.98\times10^{24}}{(3.84\times10^{8})^2}$$

= 1.99 x 10^{20} N

Similarly gravitational force on the moon due to Sun,

$F_{ms} = 4.34\times10^{20} N$

The two forces are right angles, so

$$F = \sqrt{(1.99\times10^{20})^2 + (4.34\times10^{20})^2}$$

= 4.77 x 10^{20} N.

8. Acceleration due to gravity

$$g = \frac{Gm_e}{r^2}$$

= 9.77 m/s^2

Terminal Questions:

1. 6.4 x 10^6 m

2. 4.2 x 10^{-8} N

3. 1.7 ms^{-2}

4. It is (3.6 x 10^7 m + 6.4 x 10^6 m) = 4.24 x 10^7 m from the centre of the Earth. The force is F = Gm_1m_2/r^2 = (6.67 x 10^{-11} x 6.0 x 10^{24} x 250)/ (4.24 x $10^7)^2$= 56N

5. F = Gm_1m_2/r^2,

F = G x 2 x 10^{30} x 6 x 10^{24}/ [1.5 x $10^{11}]^2$ = 6.7 x 10^{11} N

6. $m_2 = Fr^2/Gm_1$ = (4.4 x 10^{20} x (1.5 x $10^{11})^2$)/(6.67 x 10^{-11} x 2.0 x 10^{30}) = 7.4 x 10^{22} kg
7. F = Gm1m2/r2
   F = G x 100 x 80 / 0.52 = 2.14 x 10-6 N.
   This is a very small force but it does increase as the people get closer together!
   Actually this example is not accurate because Newton's law really only applies to spherical objects, or at least objects so far apart that they can be effectively considered as spherical.
8. Greater
9. Less
10. ¼
11. m_1 = 75 kg
    m_2 = 55 kg
    F = 2.25 E -8 N

d =?
$d = \text{sqrt}\,[(Gm_1m_2)/F]$
$d = \text{sqrt}\,\{[6.67\ E\ -11\ (N*m^2/kg^2][75\ kg][55\ kg]/\ 2.25\ E\ -8\ N\ \}$
d = 3.5 m

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UNIT 12: ESCAPE VELOCITYAND GRAVITATIONAL POTENTIAL

Structure

12.1 INTRODUCTION

Physics gives us the rules for how to analyse day to day how long it takes for something to fall back down to the earth. If something is thrown straight up into the air, then we can use the rules formation. If fired at an angle, the rules about motion or projectile motion help us to calculate when the object will return to the ground. But, what if the object never comes back down? This seems to fly in the face of the common sense saying "What goes up must come down," but that is because everything in our common sense experience can be disobeyed with enough effort. If you can hit escape velocity, then you are able to break free of the pull of gravity.

On Earth we always have the force of gravity acting on us. When we are above the Earth's surface we have potential or stored energy. This is called gravitational potential energy. The amount of gravitational potential energy of object on Earth depends on its mass and height above the ground. For example Books on a shelf have gravitational potential energy.

All we know, as the adventurer ship falls, its gravitational potential energy is transmitted into kinetic energy. At the bottom of the swing it is travelling at its highest speed. As it swings back up the other side it slows down as its kinetic energy is transported back into gravitational potential energy. If the mass of the adventurer ship is doubled, the kinetic energy also doubles, however if its speed is doubled its kinetic energy quadruples.

In this unit, we shall first understand what we mean by escape velocity. We shall learn how to describe escape velocity on the surface of earth. We shall learn the orbital velocity on the surface of earth or any other planet or moon. We will discuss how to calculate the gravitation potential and field due to a thin spherical shall outside or inside and on the surface of the shell. Also, we will study gravitational potential, gravitational potential energy and field due to a solid sphere at different points of the sphere. We will discuss many solved examples related to escape velocity, orbital velocity, gravitational potential and field. We will be familiar with the geostationary orbit, motion of satellite and conditions for launching the artificial satellites in space.

12.2 OBJECTIVES

After studying this unit, you should be able to -

- define escape velocity
- define orbital velocity
- apply orbital velocity to find the orbital speed of the satellite
- solve problems using escape velocity and orbital velocity
- solve the gravitational potential at different points of spherical shell
- solve the gravitational potential at different points of solid sphere
- solve the gravitational field at different points of solid sphere
- calculate the time period of the satellite

12.3 ESCAPE VELOCITY

Escape velocity is the minimum speed needed for an object to escape from the gravitational attraction of a massive body. The escape velocity from Earth is about 11.2km/s at the surface. Moreover, we can define; escape velocity as the speed at which the sum of an object's kinetic energy and its gravitational potential energy is equal to zero. For given escape velocity perpendicular to a massive body, the object will move away from the body, slowing forever and approaching but never reaching zero speed. Once escape velocity is achieved, no further impulse need be applied for it to continue its escape. In other words, if given the escape velocity, the object will move away from the other body, continually slowing and will asymptotically approach zero speed as the object's distance approaches infinity, never to return.

By Newton's law of gravitation, the force acting on a body of mass m and that of earth of mass M at a distance x is given by $F = G\frac{mM}{x^2}$

Hence, the work done by the body against the gravitational field, when it moves a distance dx in the upward direction is given by

$$dW = G\frac{mM}{x^2}.dx$$

Therefore, the total work done by the body to escape away from the Earth surface is

$$W = \int_R^\infty G\frac{mM}{x^2}.dx$$

Where, R is the radius of the Earth. Solving this integral, we get

$$W = G\frac{mM}{R}$$

We define the initial velocity of the body by v_e; the initial kinetic energy of the body of mass m is defined as

$$=\frac{1}{2}mv_e^2$$

By conservation of energy, this must be equal to the work done by it during the escape.

Therefore, in equilibrium, we can write

$$\frac{1}{2}mv_e^2 = \frac{GmM}{R}$$

On solving we get

$$v_e^2 = \frac{2MG}{R}$$

Hence $v_e = \sqrt{2MG/R}$

We know $g = \frac{MG}{R^2}$

So, from these equations we get

$$v_e = \sqrt{\frac{2MG}{R^2}R}$$

Or, solving

$$v_e = \sqrt{2gR}$$

From these equations we conclude the following:

- The escape velocity is independent of the mass of the object.
- If the radius of the planet or star of fixed mass M gets very small, the escape velocity of an object at its surface can exceed the speed of light. This will happen if the radius goes below a critical radius, R_c , given by: $R_c = \frac{2MG}{c^2}$
- When an object collapses below its critical radius, it becomes a black hole, from which nothing, not even light, can escape. The critical radius for the sun is about 3 km, while the critical radius for the earth is about 5cm.

12.3.1 Escape Velocity on Earth Surface

We have the expression for escape velocity $v_e = \sqrt{2gR}$,on putting the values of g and R, that is $g=9.8 m/s^2$ and $R= 6.4 \times 10^6$ m

$$v_e = \sqrt{2gR} = \sqrt{2\times 9.8\times 6.4\times 10^6}$$

$=11.2 \times 10^3$ m/s

Or, 11.2 km/s

So, as with surface gravity, a simple Physics equation can be used to calculate the escape velocity for a body, in this case the Earth. If you know the mass of the body and its radius v_e can be estimated. The assumption in using this formula is that the body is spherical, but this is a pretty good assumption. If the radius of a body at its equator and pole are very different, then the escape velocity is different at those places and should be calculated separately.The escape velocity for the Earth is therefore 11.2 kilometres per second. This is the velocity that an object or gas molecule needs at the surface of the Earth to be able to overcome the gravitational attraction of the Earth and escape to space.

12.3.2 Escape Velocity on Moon Surface

We know the value of g at moon is 1/6th of the Earth, radius of the moon is 1737400 m, so putting these values in equation $v_e = \sqrt{2gR}$,

We get $v_e = \sqrt{2gR} = \sqrt{2 \times (9.8/6) \times 1737400}$

On solving we get

$v_e = 2.4 km/s$

This is the escapes velocity on the surface of the moon.

12.4 ORBITAL VELOCITY

The velocity with which a satellite revolves round a planet is called orbital velocity. Consider a satellite of mass m, moving around the earth in an orbit of radius R. let M be the mass of the earth, v_0 be the orbital velocity of the satellite. We know, the necessary centripetal force, F_c acting on the satellite is provided by the gravitational force of attraction on the satellite. Gravity supplies the necessary centripetal force to hold a satellite in orbit about the earth. The circular orbit is a special case since orbits are generally ellipses, or hyperbolas in the case of objects which are merely deflected by the planet's gravity but not captured. Setting the gravity force from the universal law of gravity equal to the required centripetal force yields the description of the orbit.

In equilibrium, the gravitational force= Centripetal force,

Let m be the mass of satellite orbiting around the earth

Therefore, $F_g = F_c$,

Or, $\frac{mv_0^2}{R^2} = \frac{GMm}{R^2}$

On solving we get, $v_0 = \sqrt{\frac{GM}{R}}$, we know from the previous article,

$gR^2 = GM$

Therefore we get, $v^2 = gR$

Or, $v = \sqrt{gR}$

This is defined as the orbital velocity

$v_o = \sqrt{gR}$

Putting the value of g and R in this expression, we get

$v_o = \sqrt{9.8 \times 6.4 \times 10^6}$

$\simeq 8 km/s$

This is the required orbital velocity to launch a satellite in space.

12.4.1 The Condition for Launching of artificial Satellites

- When $v_0 = \sqrt{\frac{GM}{R}}$, satellite goes in circular orbit.
- When $v_0 > \sqrt{\frac{2GM}{R}}$ but less than the escape velocity $v_0 = \sqrt{\frac{2GM}{R}}$ then satellite goes in an elliptical orbit.
- When $v_0 = \sqrt{\frac{2GM}{R}}$ the satellite takes a parabolic path and escape to infinity.
- When $v_0 > \sqrt{\frac{2GM}{R}}$ the satellite takes a hyperbolic path and escape to infinity.

12.4.2 Time Period of the Satellite

The orbital velocity of the satellite is given by

$$v_0 = \sqrt{\frac{GM}{r}}$$

If ω is the angular velocity of the satellite, then $v_0 = r\omega$

Comparing these equations we get

$$r\omega = \sqrt{\frac{GM}{r}}$$

Or, $\omega = \sqrt{\frac{GM}{r^3}}$

By definition $\omega = 2\pi/T$

Where T is the time period of the satellite

Therefore

$$2\pi / T = \sqrt{\frac{GM}{r^3}}$$

Or, $$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

$$\frac{T^2}{r^3} = \frac{4\pi^2}{GM}$$

If, h is the height of the satellite from the surface of the earth and R tends to radius of the earth, then

$$r = R + h$$

Thus we have

$$T = 2\pi\sqrt{\frac{(R+h)^3}{GM}}$$

As we know

$$GM = gR^2$$

We get, $$T = 2\pi\sqrt{\frac{(R+h)^3}{gR^2}}$$

If satellite is very near to the surface of the earth, h ≪ R

We get

$$T = 2\pi\sqrt{\frac{R}{g}}$$

This is the expression for the time period of satellite orbiting around the earth.

12.4.3 Geosynchronous Orbit

A satellite that appears to be fixed at a position above a certain distance from earth having same period of rotation as that, of earth is called geostationary satellite.

Communication satellites can be placed in equatorial orbits at a distance which results in an orbital period of one day. Thus the satellite occupies a stationary position above the surface of the earth. To determine the height of this orbit, we simply equate the centripetal force due to gravity with the centrifugal force resulting from motion along the circular path.

12.4.4 The Motion of Satellite

A satellite is often thought of as being a projectile which is orbiting the Earth. But how can a projectile orbit the Earth? Does not a projectile accelerate towards the Earth under the influence of gravity? And as such, would not any projectile ultimately fall towards the Earth and collide with the Earth, thus ceasing its orbit? These are all good questions and represent hesitant blocks for many students of physics. We will discuss each question here. First, an orbiting satellite is a projectile in the sense that the only force acting upon an orbiting satellite is the force of gravity. Most Earth-orbiting satellites are orbiting at a distance high above the Earth such that their motion is unaffected by forces of air resistance. Indeed, a satellite is a projectile.Second, a satellite is acted upon by the force of gravity and this force does accelerate it towards the Earth. In the absence of gravity a satellite would move in a straight line path tangent to the Earth. In the absence of any forces whatsoever, an object in motion, such as a satellite, would continue in motion with the same speed and in the same direction.

This is the law of inertia. The force of gravity acts upon a high speed satellite to deviate its trajectory from a straight-line inertial path. Indeed, a satellite is accelerating towards the Earth due to the force of gravity. When launched at this speed of orbital velocity, the projectile will fall towards the Earth with a trajectory which matches the curvature of the Earth. As such, the projectile will fall around the Earth, always accelerating towards it under the influence of gravity, yet never colliding into it since the Earth is constantly curving at the same rate. Such a projectile is an orbiting satellite.

Examples 1: What is the speed required for an object like a rocket to overcome the force of earth’s gravity? Is this called exit velocity?

Solution:Well, You are right that this is sometimes called exit velocity. It is also sometimes called escape velocity. The escape velocity from the Earth's surface is about 11.2 km/sec,if we ignore air resistance. This is the speed with which you would have to launch something from the surface of the earth if you wanted it to completely escape from the earth’s gravitational pull. Since most rockets fire their engines for quite a long time (until the rocket is already rather far from the Earth’s surface), there is actually no need for the kind of rockets we are used to ever go this fast.

Examples 2: Determine the mass of Mars, where the escape velocity at mars is 5 x 10^3 m/s. Radius of the Mars = 33.97 x 10^5 kg, G= 6.67 x 10^{-11} Nm/kg^2

Solution: We know the escape velocity is given by the formula

$$v_e = \sqrt{2MG / R}$$

$$M = \frac{v_e^2 R}{2G}$$

Therefore, on putting the value in the formula

$$M = \frac{v_e^2 R}{2G} = \frac{(5\times10^3)^2 \times 33.97\times10^5}{2\times6.67\times10^{-11}}$$

$$M = 6.4\times10^{23}\,kg$$

Example 3: Calculate the escape velocity at the moon if it's Mass is 7.35 $\times 10^{22}$ Kg and radius is 1.7 $\times 10^6$ m.
Solution: given that M = 7.35 × 10^{22} Kg,
R = 1.7 × 10^6 m
Hence Escape Velocity is,

$$v_e = \sqrt{2MG/R}$$

$$v_e = \sqrt{2MG/R} = \sqrt{\frac{2\times6.67\times10^{-11}\times7.35\times10^{22}}{1.7\times10^6}}$$

$$v_e = 2.4\times10^3\,m/s$$

Self Assessment Question (SAQ) 1:A satellite is orbiting the earth. Which of the following variables will affect the speed of the satellite?

a. mass of the satellite

b. height above the earth's surface

c. mass of the earth

Self Assessment Question (SAQ) 2:Use the information to calculate the T^2/R^3 ratio for the planets about the Sun, the moon about the Earth, and the moons of Saturn about the planet Saturn. The value of G is 6.673 x 10^{-11} Nm^2/kg^2.

$$M_{sun} = 2.0\times10^{30}\,kg\,,\; M_{Earth} = 6.0\times10^{24}\,kg\,,\; M_{saturn} = 5.7\times10^{26}\,kg$$

a. T^2/R^3 for planets about Sun

b. T^2/R^3 for the moon about Earth

c. T^2/R^3 for moons about Saturn

Self Assessment question (SAQ) 3: One of Saturn's moons is named Mimas. The mean orbital distance of Mimas is 1.87 x 10^8 m. The mean orbital period of Mimas is approximately 23 hours. Use this information to estimate a mass for the planet Saturn.

Self Assessment question (SAQ) 4:Consider a satellite which is in a *low* orbit about the Earth at an altitude of 220 km above Earth's surface. Determine the orbital speed of this satellite. Use the information given below.G = 6.673 x 10^{-11} Nm^2/kg^2

$$M_{earth} = 5.98\times10^{24}\,kg\,,\; R_{earth} = 6.67\times10^6\,m$$

Self Assessment Question (SAQ) 5: Suppose the Space Shuttle is in orbit about the earth at 400 km above its surface. Use the information given in the previous question to determine the orbital speed and the orbital period of the Space Shuttle**.**

12.4GRAVITATIONAL POTENTIAL AT A POINT DISTANCE r FROM A BODY OF MASS m

Les us consider a mass m at a point O and a unit mass at a point at P. by Newton's law the force of attraction on the unit mass due to mass m is

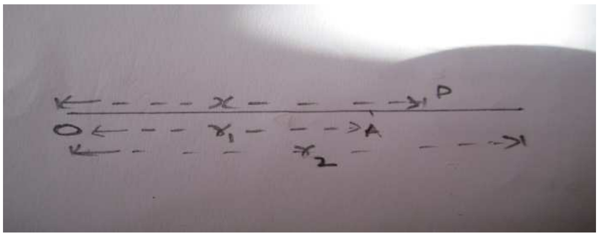

Figure 12.1

$$F = G\frac{m \times 1}{x^2}$$

$$F = G\frac{m}{x^2}$$

x is the distance of P from the point O, the direction of force is towards the point O.

The work done when the unit mass moves through the small distance dx towards O is given by

$$dW = G\frac{m}{x^2}dx$$

The total work done when it moves from B to A is given by

$$= G\int_B^A \frac{m}{x^2}dx$$

$$= Gm\int_B^A \frac{1}{x^2}dx$$

$$= Gm\left[-\frac{1}{x}\right]_B^A$$

Or, we get

$$= -Gm\left[\frac{1}{A} - \frac{1}{B}\right]$$

Given that r_1 rand r_2 are the distances of A and B from O.

Therefore it becomes

$$= -Gm\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

Suppose the B point is at infinity, we take $r_2 \to \infty$, the eq. becomes

$$= -Gm\left[\frac{1}{r_1} - \frac{1}{\infty}\right]$$

$$= -Gm / r_1$$

Or, more general, the potential difference between A and $\infty = -Gm / r$

We know the potential between A and infinity is equal to the potential at A, because the gravitational force at ∞ due to m is zero, so the work done in moving a mass from ∞ to A is also zero. Or, we can say that the potential at infinity is zero. Thus, we can say that the gravitational potential at A due to the mass m is

$$V = -Gm / r$$

It is noticed that the value of gravitational potential at an infinite distance from the mass m is zero, and it goes on decreasing as we approach that attracting mass.

Example 4: Calculate the height of the geostationary satellite from the surface of the earth.

Solution: the satellite rotates in the plane of equator with the same rotation as that of earth and thus appears to be stationary to an observer on the earth.

The angular velocity of the geostationary satellite is given by $T = 2\pi / \omega$, where T is the time period of geo stationary satellite

We know T=24 hours

Or = 86000 sec

Therefore $\omega = 2\pi / 86400$

$$= 7.3 \times 10^{-5} rad / \sec$$

By definition orbital velocity is defined as $v_0 = \sqrt{\frac{GM}{r}}$

Also we have $v_0 = r\omega$

Therefore, on simplifying, we get

$$r\omega = \sqrt{\frac{GM}{r}}$$

Or, $r = \left(\frac{GM}{\omega^2}\right)^{1/3}$

We have $G = 6.67\times10^{-11} Nm^2/kg^2$,

$M = 5.98\times10^{24} kg$

And $\omega = 7.3\times10^{-5} rad/\sec$

Putting these values in the above equation $r = \left(\frac{GM}{\omega^2}\right)^{1/3}$ we get

$r = 42\times10^6 m$

We have

$h = r - R$

Putting the values of R ($R = 6.4\times10^6 m$) and r, we get

$h = 35.6\times10^6 m$

Example 5: Why, There is no atmosphere at Moon?

Solution: Moon is about one quarter the size of Earth and it has about one-sixth of the Earth's gravity. Its gravitational pull is very weak. So escape velocity is very low. All the gas molecules on its surface, when it was formed had sufficient energy(more than the escape velocity on moon surface) to escape from its surface. That is why; there is no atmosphere on the moon's surface.

Self Assessment Question (SAQ)6: suppose you want to place a 1000 kg weather satellite into a circular orbit 300 km above the earth surface.

(a) What speed, period and radial acceleration must it have?

(b) How much work has to be done to place this satellite in orbit?

(c) How much additional work would have to be done to make this satellite escape the earth? The earth radius is $R = 6.3\times10^3 km$ and mass of the earth $M = 5.97\times10^{24} kg$

12.6 GRAVITATIONAL POTENTIAL DUE TO A SPHERICAL SHELL

Gravitational potential at a point is the potential energy that a unit mass would possess at the point in the gravitational field.

Potential difference between two points in terms of electric field, say A and B is defined as $dV = -Edr$

Also, gravitational field which is the negative potential gradient at a point is, $V = -\frac{Gm}{r}$

And gravitational potential energy of a body at a point, $= -\frac{Gm_1m_2}{r}$, where m_1, m_2 are the masses of the two bodies at a distance r.

12.6.1 At a Point Outside the Shell

Consider a thin Spherical shell of Mass M radius R and uniform density ρ. We will calculate the gravitational potential at P a distance r from the centre O of the spherical shell of radius R. We join the point O and P and cut the slice in the form of ring shown as CEFD, that is in the form of two planes close to each other and perpendicular to the radius OA, which meets the shell in C, D, E and F respectively.

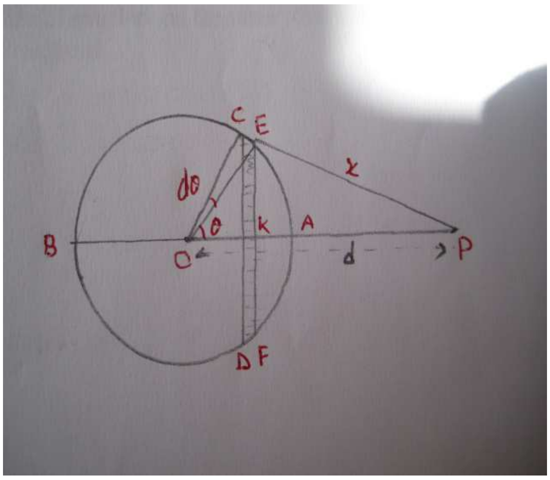

Figure 12.2

Let us take $\angle EOP = \theta$ and small angle $\angle COE = d\theta$

The radius of the ring is

$EK = OE \sin\theta$

$EK = R\sin\theta$

Therefore its circumference

$= 2\pi R\sin\theta$ and its width

$CE = Rd\theta$

Therefore, the area of the ring = circumference of the ring $\times$width

$= 2\pi R \sin\theta \times Rd\theta$

Therefore its mass is $= 2\pi R \sin\theta \times Rd\theta \times \rho$

Let the distance from P to E is x, therefore every point of the ring or slice is at a distance r from the point P. Therefore the potential due to this small slice at a point P is

$dV = mass.G / x$

$dV = (2\pi R \sin\theta \times Rd\theta \times \rho)G / x$

Now in triangle OEP, $x^2 = R^2 + r^2 - 2Rr\cos\theta$

Differentiating this equation, we get

$2xdx = 0 + 0 + 2Rr\sin\theta d\theta$

Therefore we get

$x = 2Rr\sin\theta d\theta / 2dx$

Putting this value in the definition of potential, we get $dV = -2\pi R^2 \sin\theta d\theta\rho.Gdr / Rr\sin\theta d\theta$

Or we get $dV = -2\pi R\rho Gdx / r$

On integrating it from the limit $x = r - R$ to $x = r + R$, we get the potential due to whole shell at a point P $V = \int_{r-R}^{r+r} \frac{2\pi R\rho Gdx}{r}$

$$V = \frac{2\pi R\rho G}{r}\int_{r-R}^{r+r} dx$$

Or on solving we get

$$V = \frac{2\pi R\rho G}{r}\left[x\right]_{r-R}^{r+R} = -\frac{2\pi R\rho G}{r}(2R)$$

$$V = -\frac{4\pi R^2\rho G}{r}$$

$$V = -\frac{MG}{r}$$

Therefore the potential at the point P due to the whole shell is equal to

$$V = -\frac{MG}{r}$$

This is the same potential as it would be due to a mass M at O.

Thus we can conclude that the mass of the whole shell thus behaves as though it were concentrated at its centre, in evaluating the potential outside the shell.

12.6.2 Gravitational Potential at the Surface of the Sphere

Let us consider a point P on the surface of the shell, we get the gravitational potential on integrating the expression of dV between the limit $x = 0, x = 2R$

Now, integrating the integral $V = \int_0^{2R} \frac{2\pi R\rho G dx}{r}$

We get, $V = -\frac{2\pi R\rho G}{r}(x)_0^{2R}$

Or, $V = -\frac{2\pi R\rho G}{r}(2R-0)$, in this case $r = R$, and $M = 4\pi R^2 \rho$

So we get

$$V = -\frac{M}{R}G$$

Meaning that the whole mass of the shell behaves as still it is concentrated at its centre.

12.6.3 Gravitational Potential inside the Shell of the Sphere

Let us imagine the point P inside the shell, as we have the potential due to the ring CEFD

$$dV = -\frac{2\pi R\rho G}{r}dx$$

Let us define the limit, $x = R - r, x = R + r$

Therefore the gravitational potential inside the shell is

$$V = \int_{R-r}^{R+r} -\frac{2\pi R\rho G}{r}dx$$

$$V = -\frac{2\pi R\rho G}{r}(x)_{R-r}^{R+r}$$

Or, $V = -\frac{2\pi R\rho G}{r}(R + r - R + r)$

We get,

$$V = -\frac{2\pi R\rho G}{r}(2r)$$

$V = -2\pi R\rho G$

Again since $M = 4\pi R^2 \rho$,

Therefore on adjusting, we get

$V = -\frac{M}{R}G$

It is same as the potential on the surface of the shell

It's means that the potential remains the same inside the shell and is equal to the value of the potential on the surface of the shell.

12.7 GRAVITATIONAL POTENTIAL AND GRAVITATIONAL FIELD DUE TO A SOLID SPHERE

12.7.1 Point P is Outside the Sphere:

Consider a solid sphere of mass M, radius R and uniform density (we assumed that the mass is uniformly distributed) ρ. Let us consider it is made up of large number of thin spherical shells whose radii vary from O to R. let r be the radius of one such shell and dr its thickness. Let the point P be at a distance x from the centre

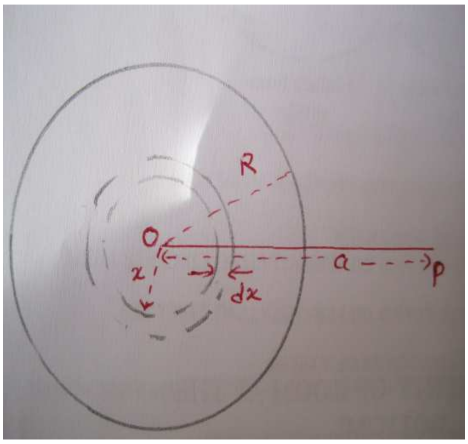

Figure 12.3

Mass of the shell is defined as= area of the shell x density of the shell

$= 4\pi r^2 dr\rho$

Therefore by definition, the gravitational potential at a point P due to the shell is

$$dV = -\frac{Gm}{x} = G(4\pi r^2 dr\rho)/x$$

Therefore the potential due to the entire sphere is

$$V = -\int_0^R G(4\pi r^2 dr\rho)/x$$

$$V = -\frac{G4\pi\rho R^3}{3x}$$

Or it may be written as $V = -\frac{GM}{x}$, where $M = (4/3)\pi R^3 \rho$

Gravitational Field:

By definition the gravitational field is defined as $E = -\frac{dV}{dx}$

Therefore

$$E = -\frac{d}{dx}\left(-\frac{GM}{x}\right)$$

$$E = -\frac{GM}{x^2}$$

This is the electric field at a distance x; its unit is volt/m^2.It has a great historical importance, because it allowed Newton to apply his law of gravitation to the motion of the moon.

12.7.2 Point P on the Surface of Sphere

We have derived the gravitational Potential outside the sphere**,**

$$V = -\frac{GM}{x}$$

Therefore, When $x = R$

Gravitational potential on the surface of sphere is

$$V = -\frac{GM}{R}$$

Also $E = -\frac{dV}{dx} = -\frac{GM}{x^2}$, so at the surface (x=R), gravitational field $E = -\frac{GM}{R^2}$

12.7.3 Point P Inside the Sphere:

The solid sphere may be imagined to be made up of inner solid sphere of radius b, surrounded by a number of hollow spheres or spherical shells, concentric with it and with their radii ranging from b to R. the potential at P due to solid sphere is equal to the sum of the potential at P due to the inner sphere plusall such spherical shells outside it.

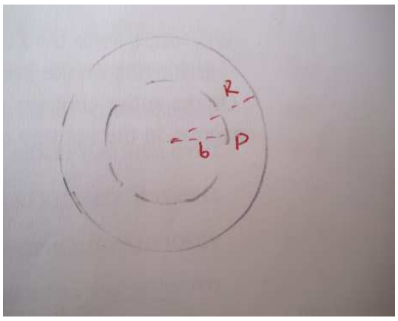

Figure 12.4

Consider point P inside the sphere of radius R on an imaginary spherical surface of radius b. the gravitational potential at P due to all shells between R and b is

$$V_1 = -\int_b^R G(4\pi x^2 dx\rho)/x$$

Mass of shell is defined as $= 4\pi x^2 dx\rho$

$$V_1 = -G4\pi\rho \int_b^R x dx$$

Or, $V_1 = -G4\pi\rho \left[\frac{x^2}{2} \right]_b^R$

$$V_1 = -G4\pi\rho \left[\frac{R^2}{2} - \frac{b^2}{2} \right]$$

Or, $V_1 = -G4\pi\rho \left[\frac{R^2 - b^2}{2} \right]$

The Potential at P due to the shells with in the sphere of radius b is

$V_2 = -Gm/b$, where m be the mass of sphere of radius b

Where $m = \frac{4}{3}\pi b^3 \rho$

Therefore the gravitational potential at P due to the entire sphere is

$$V = V_1 + V_2$$

Putting the values in this equation, we get

$$V = -\frac{G4\pi\rho}{2}(R^2 - b^2) - \frac{4G\pi\rho}{3}b^2$$

On solving

$$V = -G4\pi\rho\left(\frac{R^2}{2} - \frac{b^2}{2} - \frac{b^2}{3}\right)$$

Or, $V = -G4\pi\rho\left(\frac{3R^2 - b^2}{6}\right)$

Or $V = -\frac{G4\pi\rho}{6}(3R^2 - b^2)$

Or in terms of the mass of sphere

$$V = -\frac{GM}{2R^3}(3R^2 - b^2)$$

Where M is the total mass of sphere, which is given by

$$M = \frac{4}{3}\pi R^3 \rho$$

Gravitational Field:

Gravitational Field is defined as

$$E = -\frac{dV}{db}$$

Therefore,

$$E = -\frac{d}{db}\frac{GM}{2R^3}(3R^2 - b^2)$$

Or, $E = \frac{GM}{2R^3}(0 - 2b)$

Or, $E = \frac{GM}{2R^3}(-2b)$

Or we get, $E = -\frac{GMb}{R^3}$

That is the intensity of the field is directly proportional to the distance from the centre of the sphere.

Clearly, therefore at the centre of the sphere b is zero, so the intensity of gravitational field is zero.

Or, we can say that the intensity of the gravitational field inside a solid sphere has its maximum value at its centre.

12.8 SUMMRRY

In this unit, you have studied escape velocity and orbital velocity. As you move away, Earth's gravity would try to slow you down. However, the pull of gravity gets weaker and weaker. At escape speed, gravity would never be able to bring your speed to zero. You had moved away from Earth forever. At orbital velocity, you no longer move away from Earth, you stay up at the same distance. We know, real orbits are ellipses so that you move up and down. But you keep the same average distance. Escape speed from Earth's surface is 11.2 km/s. If you start from higher, then escape speed is less, because gravity gets weaker as you move away. We have studied that, when a satellite moves in a circular orbit, centripetal acceleration is provided by the gravitational acceleration of the earth.

Any two masses whether or not Earth and the moon, experience a mutual gravitational force that tends to pull them together. A mass in a position to be pulled to another position by a gravitational force has gravitational potential energy. We have studied that the gravitational potential energy of two masses separated by a distance is inversely proportional to the distance between them. The potential energy is never positive, it is zero only when the two bodies are infinitely far apart.

We have discussed the gravitational potential, gravitational energy and gravitational field due to a spherical shell and solid sphere. Usually, someone has to do work to get the object to the elevated position. Also, many live examples are discussed to clear the concepts of escape velocity, orbital velocity, gravitational potential, gravitational field and gravitational potential energy. Many solved examples are given in the unit to make the concepts clear. To check your progress, self assessment questions (SAQs) are given place to place.

12.9 GLOSSARY

1. Adventure	exploration
2. Transmitted	communicated
3. Transport	carrying
4. Quadruples	multiplies
5. Critical	serious
6. Assumptions	suppositions
7. Geostationary	constant with Earth
8. Elliptical	oval

9. Infinity endlessness

10. Communication message

11. Projectile missile

12. Hesitate delay

13. Launched threw

14. Uniformity consistency

12.10 TERMINAL QUESTIONS

1. What is the gravitational potential energy of the moon with respect to the earth? The mass of the moon is 7.35×10^{22}kilograms and the mass of the earth is 5.98×10^{24} kilograms. The earth moon distance is 384 400 kilometers.
2. A satellite of mass 1000 kilograms is launched with a speed of 10 km/sec. It settles into a circular orbit of radius 8.68×10^{3}km above the centre of the earth. What is its speed in this orbit? (M_e= 5.98×10^{24} and r_e = 6.38×10^{6} m).
3. Determine the orbital speed of the International Space Station - orbiting at 350 km above the surface of the Earth. The radius of the Earth is 6.37 x 10^6 m. (GIVEN: M_{Earth} = 5.98 x 10^{24} kg)
4. Determine the orbital speed of the Earth as it orbits about the Sun. (GIVEN: M_{sun} = 1.99 x 10^{30} kg and Earth-sun distance = 1.50 x 10^{11} m)
5. In 2009, NASA's Messenger spacecraft became the second spacecraft to orbit the planet Mercury. The spacecraft orbited at a height of 125 miles above Mercury's surface. Determine the orbital speed and orbital period of Messenger. (GIVEN: $R_{Mercury}$ = 2.44 x 10^6 m; $M_{Mercury}$ = 3.30 x 10^{23} kg; 1 mi = 1609 m)
6. Geostationary satellites are satellites which are orbiting the Earth above the equator and make one complete orbit every 24 hours. Because their orbital period is synchronized with the Earth's rotational period, a geostationary satellite can always be found in the same position in the sky relative to an observer on Earth. (GIVEN: M_{earth} = 5.98 x 10^{24} kg)
   a. Determine the orbital radius of a geostationary satellite.
   b. Determine the orbital speed of a geostationary satellite.
   c. Determine the acceleration of a geostationary satellite
7. A satellite wishes to orbit the earth at a height of 100 km above the surface of the earth. Determine the speed, acceleration and orbital period of the satellite. (Given: M_{earth} = 5.98 x 10^{24} kg, R_{earth} = 6.37 x 10^6 m)

12.11 ANSWERS

Self Assessment Questions (SAQ):

1. b and c

As seen in the equation $v_0 = \sqrt{\dfrac{GM}{r}}$,the mass of the central body (earth) and the radius of the orbit affect orbital speed. The orbital radius is in turn dependent upon the height of the satellite above the earth.

2.For each case, use the equation $T^2/r^3 = \frac{4\pi}{GM}$

a. Sun $T^2/R^3 = 2.96 \times 10^{-19}$
b. Earth $T^2/R^3 = 9.86 \times 10^{-14}$
c. Saturn $T^2/R^3 = 1.04 \times 10^{-15}$
(All answers in units of s^2/m^3.)

3. Using the T and R values given, the T^2/R^3 ratio is 1.05×10^{-15}. This ratio is equal to $T^2/R^3 = \frac{4\pi}{GM}$. Using the G value and the calculated ratio, the mass of Saturn can be found to be 5.64×10^{26} kg.

4. The orbital speed can be found using $v_0 = \sqrt{\frac{GM}{R}}$. The R value (radius of orbit) is the earth's radius plus the height above the earth - in this case, 6.59×10^6 m. Substituting and solving yields a speed of 7780 m/s.

5.

The orbital speed can be found using $v_0 = \sqrt{\frac{GM}{R}}$,The R value (radius of orbit) is the earth's radius plus the height above the earth - in this case, 6.77×10^6 m. Substituting and solving yields a speed of 7676 m/s.

6. First we find the radius of the orbit of the satellite, it is r= 6380+300=6680km

Or, r= 6.68×10^6m.

We have $v_0 = \sqrt{\frac{Gm}{r}}$, where m is the mass of satellite

$$v_0 = \sqrt{\frac{6.67\times10^{-11}\times5.97\times10^{24}}{6.68\times10^{6}}}$$

$$v_0 = 7720m/s$$

The time period is given by

$$T = 2\pi r/v$$

$$T = 2\pi\times6.68\times10^6/7720$$

$$T = 5440s$$

The radial acceleration is given by $a_{rad} = \frac{v^2}{r}$

Putting the values in this equation we get

$$a_{rad} = \frac{v^2}{r} = \frac{(7720)^2}{6.68\times10^6}$$

$$a_{rad} = 8.92 m/s^2$$

(b) The work required is the difference between E_2, the total mechanical energy when the satellite is in orbit, and E_1, the original mechanical energy when the satellite was at rest on the launch pad on earth. In orbit the energy is given by

$$E = K + U = \frac{1}{2}mv^2 + (-\frac{Gm_e m}{r})$$

$E = -\frac{Gm_e m}{2r}$, where m_e of earth and m mass of satellite

We define this energy as

$$E_2 = -\frac{Gm_e m}{2r}$$

$$E_2 = -\frac{6.67\times10^{-11}\times5.97\times10^{24}\times1000}{2\times6.68\times10^6}$$

$$E_2 = -2.99\times10^{10} J$$

At rest on the earth surface, the energy is purely potential

So, $E_1 = -\frac{Gm_e m}{r}$

$$E_2 = -\frac{6.67\times10^{-11}\times5.97\times10^{24}\times1000}{6.68\times10^6}$$

$$E_1 = -6.25\times10^{10} J$$

So, required work $W = E_2 - E_1$

Or, $W = 3.26\times10^{10} J$

(c) From part (b), the total mechanical energy must be zero for a satellite to escape to infinity. The total mechanical energy in the circular orbit is $E_2 = -2.99\times10^{10} J$, to increase this to zero, an amount of work equal to $2.99\times10^{10} J$ would have to be done. This extra energy could be supplied by rocket engines attached to the satellite.

Terminal Questions:

1. $U = -\frac{GMm}{r}$

   Putting the values we get,

   $U = -7.63 \times 10^{28} J$
2. This problem involves the conservation of energy. The initial kinetic energy is given by $1/2mv^2 = 1/2 \times 1000 \times (10000)^2 = 5 \times 10^10$ Joules. It also has some initial gravitational potential energy associated with its position on the surface

   $U = -\frac{GMm}{r}$

   $U = -6.25 \times 10^{10} J$

   The total energy is then given by $E = T + U_i = -1.25 \times 10^{10}$ Joules. In its new orbit the satellite now has a potential energy

   $U = -\frac{GMm}{r}$

   $U = -4.6 \times 10^{10} J$

   The kinetic energy is given by $T = E - U = (-1.25 + 4.6) \times 10^{10} = 3.35 \times 10^{10}$ Joules. We can easily now find the velocity: $V = \sqrt{\frac{2T}{m}} = 8.1 \times 10^3 m/s$

3. 7.69 x 10^3 m/s
4. 2.98 x 10^4 m/s
5. Speed: 2.89 x 10^3 m/s

   Period: 5.75 x 10^3 s
6. a. 4.23 x 10^7 m

   b. 3.07 x 10^3 m/s

   c. 0.223 m/s^2
7. Given,R = R_{earth} + height = 6.47 x 10^6 m

   M_{earth} = 5.98x10^{24} kg

   G = 6.673 x 10^{-11} N m^2/kg^2

   Unknown:

   V, a and T=?

The radius of a satellite's orbit can be found from the knowledge of the earth's radius and the height of the satellite above the earth. The radius of orbit for a satellite is equal to the sum of the earth's radius and the height above the earth. These two quantities can be added to yield the orbital radius. In this problem, the 100 km must first be converted to 100 000 m before being added to the radius of the earth. The equations needed to determine the unknown are listed. We will begin by determining the orbital speed of the satellite using the following equation

$$V = \sqrt{\frac{GM}{r}} = 7.8 \times 10^3 m/s$$

$$a = \frac{v^2}{r} = \frac{GM}{r^2}$$

$$a = 9.53 m/s^2$$

The period can be calculated using the following equation:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{v/r}$$

Or, $T^2 = \frac{4\pi r^3}{GM}$

We get

T=5176sec

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UNIT 13: CONSERVATIVE FORCE AND INVERSE SQUARE LAW

Structure

13.1 INTRODUCTION

As you saw when lifting a book, the work that you do against gravity in lifting is stored somewhere... Physicists say that it is stored in the gravitational field or stored in the Earth/book system and is available for kinetic energy of the book once you let go. Forces that store energy in this way are called conservative forces. Gravity is a conservative force, and there are many others. Elastic (Hooke's Law) forces, electric forces, etc. are conservative forces. As you say when pushing a book, the work that you do against friction is apparently lost - it is certainly not available to the book as kinetic energy! Forces that do not store energy are called nonconservative or dissipative forces. Friction is a nonconservative force, and there are others. Any friction-type force, like air resistance, is a nonconservative force. The energy that it removes from the system is no longer available to the system for kinetic energy. Of course, if energy is a "real thing," the energy taken away by a nonconservative force can't just disappear! I wonder where it goes!

A collision or crash is an event in which two or more bodies exert forces on each other for a relatively short time. Although the most common colloquial use of the word "collision" refers to incidents in which two or more objects collide, the scientific use of the word "collision" implies nothing about the magnitude of the force.

Collision is short-duration interaction between two bodies or more than two bodies simultaneously causing change in motion of bodies involved due to internal forces acted between them, during this event. Collisions involve forces; there is a change in velocity. The magnitude of the velocity difference at impact is called the closing speed. All collisions conserve momentum. What distinguishes different types of collisions is whether they also conserve kinetic energy. The line of impact is the line which is collinear to the common normal of the surfaces that are closest or in contact during impact.

In the early 1600s, Johannes Kepler proposed three laws of planetary motion. Kepler was able to summarize the carefully collected data of his mentor - Tycho Brahe - with three statements that described the motion of planets in a sun-centered solar system. Kepler's efforts to explain the underlying reasons for such motions are no longer accepted; the actual laws themselves are still considered for an accurate description of the motion of any planet and any satellite. Kepler's three laws of planetary motion with applications are discussed in this unit.

In this unit, we shall first understand what we mean by conservative and non-conservative forces. We shall learn how to describe the motion of a body under conservative and non-conservative forces. In this unit we shall also discuss force as gradient of potential energy. We shall discuss the elastic and non-elastic collision and application to variety of situations. We shall define the centre of mass frame and Kepler’s law with their applications to physical world.

13.2 OBJECTIVE

After studying this unit, you should be able to understand-

- What are conservative forces
- What are non-conservative forces
- Applications of conservative and non-conservative forces
- What are elastic and non-elastic collisions
- Solution of problems of elastic and non-elastic collisions
- Application of elastic and non-elastic collisions
- Kepler's law and their applications in planetary motion
- Kepler's law to solve problems

13.3 WHAT ARE CONSERVATIVE AND NON CONSERVATIVE FORCES

It is important to know the difference between conservative and nonconservative forces. The work a conservative force does on an object is path-independent; the actual path taken by the object makes no difference. Fifty meters up in the air has the same gravitational potential energy whether you get there by taking the steps or by hopping on a wheel. That's different from the force of friction, which dissipates kinetic energy as heat. When friction is involved, the path you take matters — a longer path will dissipate more kinetic energy than a short one. For that reason, friction is a nonconservative force.

For example, suppose you and some buddies arrive at a majestic peak that rises h meters into the air. You can take two ways up — the quick way or the scenic route. Your friends drive up the quick route, and you drive up the scenic way, taking time out to have a picnic and to solve a few physics problems. You pull out this equation $\Delta P = mg(h_f - h_i)$ .This equation basically states that the actual path you take when going vertically from h_i to h_f doesn't matter. All that matters is your beginning height compared to your ending height. Because the path taken by the object against gravity doesn't matter, gravity is a conservative force.

13.3.1 Conservative Forces

A conservative force is a force that acts on a particle, such that the work done by this force in moving this particle from one point to another is independent of the path taken. To put it another way, the work done depends only on the initial and final position of the particle. Two examples of conservative forces are gravitational forces and elastic spring forces.

Gravitational force is a conservative force: When we throw a ball upward against the gravity, the ball reaches a certain height coming shortly to rest so that its kinetic energy becomes zero. Subsequently starts to come down under gravity. During the down trip the downward pull of the earth provides the kinetic energy to the ball. When, it reaches the starting point, the kinetic energy becomes same, as its initial kinetic energy with which it was thrown**.**

Elastic Force is a conservative force: when a block is moved from a position on a smooth horizontal plane with a velocity so as to compress a spring. First, it is bought to rest by the elastic force of the string and loses all kinetic energy. Then the compressed spring re-expands and the block moves back under the elastic force gaining kinetic energy. As the block returns to its initial position, it gains the same velocity, thereby attains the same kinetic energy, with which it was compressed. So, we can say that the elastic force is a conservative force.

13.3.2 Non Conservative Forces

A non conservative force is a force that acts on a particle or point, such that the work done by this force in moving this particle from one point to another is dependent on the path taken. To put it another way, the work done depends on the path itself. For example, a frictional force is non conservative because the work done by friction always acts in the direction of travel and therefore depends on the length of the path taken. If we slide a stone on a rough floor between two points along different paths, the work done by the friction forces would be different.

Thus, if a system is acted on by a non conservative force such as friction, and the system returns to its original position, then that system will experience a net loss of energy, due to those forces. Energy will thus not be conserved for the system. This makes sense intuitively since we know friction is a source of energy loss. This is why we always try to minimize friction in moving parts and machine components, so as to minimize the energy wasted**.**

13.3.3 Central force is conservative:

Central force is a force which acts upon a particle and always directed towards or away from a point. The magnitude of the central force depends only on the distance of the particle from that point. Gravitational, elastic and electrostatic forces are the example of central forces.

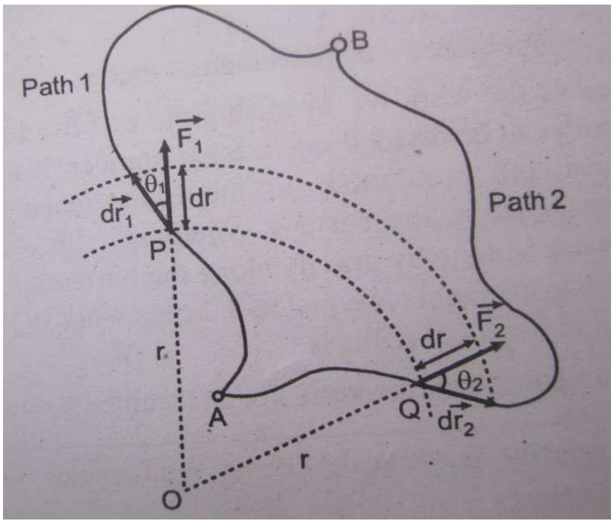

Figure 13.1

Two points A and B are connected by two random path say path1 and path 2. Consider a particle moves from A to B along any path under a central force from point O. let us draw two sectors of radii r and r+dr. consider two forces $\vec{F}_1$ and $\vec{F}_2$ acting on the particles say P and Q. let $d\vec{r}_1$ and $d\vec{r}_2$ be the displacements along the path 1 and path 2. Consider θ_1 and θ_2 be the angles between $\vec{F}_1$ and $d\vec{r}_1$, and $\vec{F}_2$ and $d\vec{r}_2$. Using vector algebra

$$\vec{F}_1.d\vec{r}_1 = F_1 dr_1 \cos\theta_1$$

And $\vec{F}_2.d\vec{r}_2 = F_2 dr_2 \cos\theta_2$

Since, P and Q are at the equal distances from O, so the magnitude of the central forces is equal that is $F_1 = F_2$. The projection of $d\vec{r}_1$ and $d\vec{r}_2$ on $\vec{F}_1$ and $\vec{F}_2$ are equal. Therefore we can write

$$\vec{F}_1.d\vec{r}_1 = \vec{F}_2.d\vec{r}_2$$

If we integrate it throughout the path from A to B , we get

$$\int_A^B \vec{F}_1.d\vec{r}_1 = \int_A^B \vec{F}_2.d\vec{r}_2$$

This shows that the work done

$W = \int_A^B \vec{F}.d\vec{r}$ is independent of the path. So we conclude that the central force is a conservative force.

We can also show that the work done around the closed path is zero.

The work done in moving a particle from A to B is given as

$W_{A\to B} = \int_A^B \vec{F}.d\vec{r}$ along the path 1, similarly the work done B to A

$$W_{B\to A} = \int_B^A \vec{F}.d\vec{r} = -\int_A^B \vec{F}.d\vec{r}$$

For conservative force work done must be same, so

$$W_{A\to B} = W_{B\to A}$$

Hence $\int_A^B \vec{F}.d\vec{r} = \int_A^B \vec{F}.d\vec{r}$

Or, $W_{A\to B} + W_{B\to A} = 0$

Thus, conservative force is one which draws or supplies no energy from or to a body in a complete round trip. A conservative force does zero total work on any closed path.

13.4 CONSEVATIVE FORCE AS NEGATIVE GRADIENT OF POTENTIAL ENERGY

Let us consider a body which is capable to do work by virtue of its position, state of strain and configuration. This capability of work is known as the potential energy. As you know that the water at the top of waterfall can rotate a turbine when falling on it. By virtue of its position, water has this ability. Moreover, a spiral clock spring may keep the clock operating by goodness of its state of strain. Both have the potential energy, the first has the gravitational potential energy and the second spiral clock has the elastic potential energy. While doing work, these are converted into kinetic energy. Thus the potential energy of a body or the system of bodies is infact a form of stored energy which can be recovered and converted into kinetic energy, which is represented by U.

When a body under conservative force such as gravitational force or elastic force is taken from one position to another, then the work done in this process is stored as potential energy in the body. The difference in the potential energy of the body at two different positions is defined as the work done in moving the body from one position to the other in the absence of frictional forces.

If a particle acted upon by a conservative force moves from a space point to another point, then the potential energy at $\vec{r}$ is given by

$U(\vec{r}) = -\int_{\vec{r}_0}^{r} \vec{F}.d\vec{r}$, where $\vec{r}_0$ refers to the position of zero potential energy.

Since, $\vec{F}.d\vec{r} = (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}).(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$\vec{F}.d\vec{r} = F_x dx + F_y dy + F_z dz$

On differentiating the equation partially, we get

$$U(\vec{r}) = -\int_{x_0}^{r} dU(x,y,z) = -\int_{x_0}^{x} F_x dx - \int_{y_0}^{y} F_y dy - \int_{z_0}^{z} F_z dz$$

On differentiating the equation partially, we get

$F_x = -\frac{dU}{dx}, F_y = -\frac{dU}{dy}, F_z = -\frac{dU}{dz}$

In vector form $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

Or, $\vec{F} = \frac{dU}{dx}\hat{i} - \frac{dU}{dy}\hat{j} - \frac{dU}{dz}\hat{k}$

Or $\vec{F} = -(\frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k})U$

Or, $\vec{F} = -\vec{\nabla}U$

Or $\vec{F} = -gradU$

Where, $\vec{\nabla} = (\frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k})$

That is conservative force is the negative gradient of potential.

Self Assessment Question (SAQ) 1: Show that the curl of conservative force is zero.

Self Assessment Question (SAQ) 2: If a force $\vec{F} = (2xy + z^2)\hat{i} + x^2\hat{j} + 2xz\hat{k}$, then show that it is conservative force.

Self Assessment Question (SAQ) 3: The position of a moving particle at an instant is given by $\vec{r} = A\cos\theta\hat{i} + A\sin\theta\hat{j}$ show that the force acting on the particle is conservative.

13.5 COLLISION

We use the term collision to represent the event of two particles coming together for a short time and thereby producing impulsive forces on each other. These forces are assumed to be much greater than any external forces present. If the force of interaction between the colliding bodies is conservative, the kinetic energy remains conserved in the collision and the collision is said to be elastic. Collisions between atomic, nuclear and fundamental particles are usually elastic.

When the kinetic energy is changed in the collision, the collision is said to be inelastic. Collisions between gross bodies are always inelastic to some extent. When two bodies stick together after collision, the collision is said to be completely inelastic. When a bullet hitting a target remains embedded in the target, the collision is completely inelastic.

13.5.1Momentum Conservation in Collision

Consider a collision between two bodies of masses m_1 and m_2. During the collision they exert forces on one another which are equal and opposite. Let $\vec{F}_{12}$ and $\vec{F}_{21}$ forces exerted on body 1 by body 2 and vice versa at any time t.

The change in momentum of body 1 resulting from the collision from time t_i to t_f is

$$\Delta\vec{p}_1 = \int_{t_i}^{t_f} \vec{F}_{12}dt \qquad (1)$$

Similarly

$$\Delta\vec{p}_2 = \int_{t_i}^{t_f} \vec{F}_{21}dt \qquad (2)$$

Since, $\vec{F}_{12} = -\vec{F}_{21}$

Therefore, $\Delta\vec{p}_1 = -\Delta\vec{p}_2$

Or we can write, $\Delta\vec{p}_1 + \Delta\vec{p}_2 = 0$

Meaning that, if there are no external forces, the total change in momentum of the system is zero.

13.5.2 Elastic Collision in one Dimension

Consider an elastic one dimensional head on collision between two bodies as shown in figure.

We have by the law of conservation of momentum

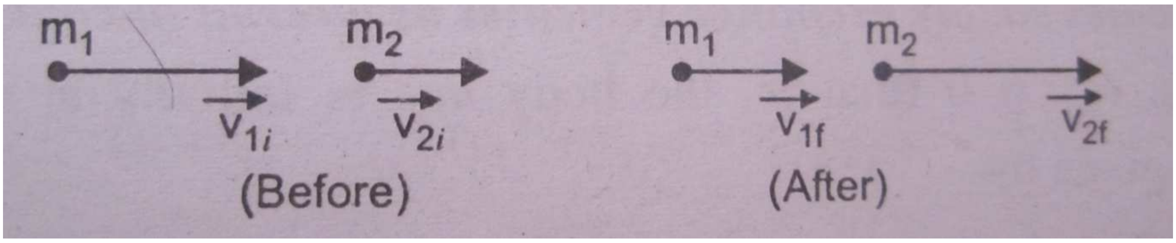

Figure 13.2

$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$,

Or, $m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i})$ (1)

Since the collision is elastic and kinetic energy is conserved. That is

$$\frac{1}{2} m_1 v^2{}_{1i} + \frac{1}{2} m_2 v^2{}_{2i} = \frac{1}{2} m_1 v^2{}_{1f} + \frac{1}{2} m_2 v^2{}_{2f}$$

Or, $m_1(v^2{}_{1i} - v^2{}_{1f}) = m_2(v^2{}_{2f} - v^2{}_{2i})$, (2)

Dividing equations (2)/(1)we get

$$\frac{(v^2{}_{1i} - v^2{}_{1f})}{(v_{1i} - v_{1f})} = \frac{(v^2{}_{2f} - v^2{}_{2i})}{(v_{2f} - v_{2i})}$$

Or we get

$v_{1i} - v_{2i} = v_{2f} - v_{1f}$, (3)

Thus in an elastic one dimensional collision, the relative velocity of attempt before collision is equal to the relative velocity of separation after collision.

We get the final velocities of the bodies from the above equation. From equation (3)

$v_{2f} = v_{1i} + v_{1f} - v_{2i}$, putting this value in equation(1), we get

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_1}{m_1 + m_2} v_{2i}$$

Similarly putting $v_{1f} = v_{2i} + v_{2f} - v_{1i}$ in equation (1), we get

$$v_{2f} = \frac{m_2 - m_1}{m_1 + m_2} v_{2i} + \frac{2m_1}{m_1 + m_2} v_{1i}$$

We discuss different cases:

Case 1. When $m_1 = m_2$ equation (1) gives

$v_{2f} - v_{2i} = v_{1i} + v_{1f}$, on comparing equation (3), we have

$v_{2i} = v_{1f}$

And $v_{1i} = v_{2f}$

So we can conclude that in one dimensional elastic collision of the two bodies of equal masses, the bodies simply exchange velocities as a result of collision.

Case 2. When $v_{2i} = 0$, means the body m_2 is initially at rest, so the final velocities are given by the following equations

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$

And

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

We may consider three conditions

(i) if $m_1 = m_2$, then $v_{1f} = 0$ and $v_{2f} = v_{1i}$. The first body is stopped and the second body takes off with the velocity the first one had initially. Now, both the momentum and the kinetic energy of the first are completely transferred to the second.

(ii) If $m_2 > m_1$ then $v_{1f} = -v_{1i}$. That is when a light body collides with a much heavier body at rest, the velocity of light body is more or less reversed and the heavier body remains at rest. A ball dropped on the earth rebounds with reversed velocity attaining the same height from which it falls, if collision is elastic.

(iii) If $m_2 < m_1$ then $v_{1f} = v_{1i}$ and $v_{2f} = 2v_{1i}$. It means when a heavy body collides with a much lighter body at rest, the velocity of the heavy body remains almost unaffected but the light body moves twice the velocity of heavy body.

Example 1: An empty freight car of mass m_1=10000kg rolls at $v_1 = 2m/s$ on a level road and collides with a loaded car of mass $m_2 = 20000kg$ standing at rest. If the car couple together, find the speed v' after the collision and also the loss in kinetic energy. What should be the speed of the loaded car toward the empty car, in order that both shall be brought to rest by the collision?

Solution**:** we have by the conservation of momentum

$$m_1 v_1 = (m_1 + m_2) v'$$

On putting the value, we get

$$v' = \frac{m_1}{m_1 + m_2} v_1$$

=0.67m/s

Kinetic energy loss=$\frac{1}{2} m_1 v^2{}_1 - \frac{1}{2}(m_1 + m_2) v'^2$

$$= \frac{4000}{3} = 13333\, jule$$

If v_2 be the speed of the loaded car towards the empty car to keep $v' = 0$. So,

$$m_1 v_1 - m_2 v_2 = 0$$

Therefore, $v_2 = \frac{m_1}{m_2} v_1$

$$v_2 = \frac{m_1}{m_2} v_1$$

$$v_2 = \frac{10000 \times 2}{20000} = 1 m / s$$

Example 2: a bullet of mass 10kg moving horizontally with speed of 500m/s passes through a block of wood of mass 1kg, initially at rest on frictionless surface. The bullet comes out of the block with a speed of 200m/s. calculates the final speed of the block.

Solution:

By conservation of momentum, we have

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}$$

0.01x500=0.01x 200+ 1x v_{2f}

On solving we get

V_{2f}=3m/s

Example 3: A particle of mass m_1 moving with velocity u_1 collides elastically with another particle of mass m_2 at rest. After collision, the two particles move with equal speeds in opposite directions. How m_1 is related to m_2.

Solution: consider v_1 and v_2 the velocities of particle m_1 and m_2 after collision. Therefore we have

$$v_1 = -v_2$$

Momentum before collision

$m_1u_1 + m_2(0)$

$= m_1u_1$

After collision

$m_1v_1 + m_2v_2$

$= m_1v_1 - m_2v_1$, using above equation

Hence, by conservation of linear momentum, we have

$m_1u_1 = m_1v_1 - m_2v_1$

Or, $v_1 = \frac{m_1}{m_1 - m_2}u_1$, by the conservation of kinetic energy, we get

$\frac{1}{2}m_1u^2{}_1 = \frac{1}{2}(m_1 + m_2)v_1{}^2$

On solving, we get

$\frac{1}{2}m_1u^2{}_1 = \frac{1}{2}(m_1 + m_2)(\frac{m_1}{m_1 - m_2})^2u_1^2$

Or, $m_1^2 + m_2^2 - 2m_1m_2 = m_1^2 + m_1m_2$

On solving we get

$m_2 = 3m_1$

13.6 CENTRAL FORCE

A central force is a force which acts towards or away from a fixed point and its magnitude depends only on the distance from a fixed point. This point is called as centre of force.

Mathematically it is represented as $\vec{F} = f(r)\hat{r}$, where $f(r)$ is a function of distance r of the particle from a fixed point. Unit vector $\hat{r}$ is a vector along radius vector $\vec{r}$ of the particle with respect to fixed point.

Examples of Central force:

- The gravitational force acting on a particle by another particle which is stationary in an inertial frame of reference is a central force, which is always directed towards the Sun.
- A particle attached to one end of a spring whose other end is stationary in an inertial frame of reference. The spring always pulls towards the fixed end.

- The electrostatic force acting on a charged particle by another. The electron in Hydrogen atom moves under a central force which is always directed towards the nucleus.

13.6.1 Centre of Mass Coordinates

Let us consider two particle masses m_1 and m_2 with the position vector $\vec{r}_1$ and $\vec{r}_2$ with respect to the origin in the laboratory frame of reference. The centre of mass position vector is as follow

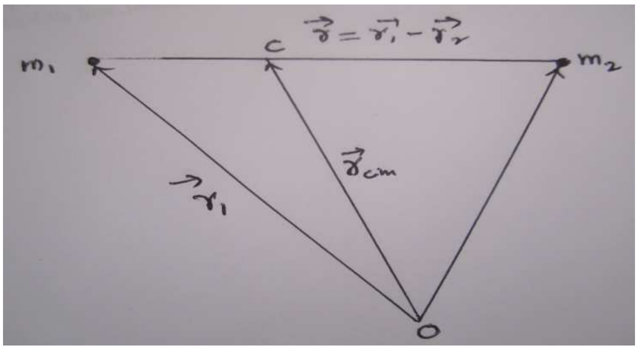

Figure 13.4

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

If $\vec{r}$ be the position vector relative to m_1 and m_2, which can be written as

$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

On solving these two equations we get

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2(\vec{r}_1 - r)}{m_1 + m_2}$$

Or may be written as

$$\vec{r}_1 = \vec{r}_{cm} + \frac{m_2}{m_1 + m_2}\vec{r}$$

We have reduced mass defined as

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Therefore the above equation written as

$$\vec{r}_1 = \vec{r}_{cm} + \frac{\mu}{m_1}\vec{r}$$

Likewise

$$\vec{r}_2 = \vec{r}_{cm} - \frac{\mu}{m_2}\vec{r}$$

$\vec{r}_{cm}$ is zero in the centre of mass coordinate system as shown in figure? Therefore

$$\vec{r}_1^{\,'} = \frac{\mu}{m_1}\vec{r}$$

$$\vec{r}_2^{\,'} = -\frac{\mu}{m_2}\vec{r}$$

These are the relation between relative and centre of mass coordinates.

Example 4: Show that the centre of mass of two particles subject to their mutual interaction moves with a constant velocity.

Solution: Let us consider two particle masses m_1 and m_2 with the position vector $\vec{r}_1$ and $\vec{r}_2$ with respect to the origin in the laboratory frame of reference. The force of interaction between them is given as $\vec{F} = F\hat{r}$, where r is the unit vector and F is the magnitude of the force between them.

Equation of motion of the particle is written as

$$m_1\frac{d^2\vec{r}_1}{dt^2} = -F\hat{r}$$

and

$$m_2\frac{d^2\vec{r}_2}{dt^2} = F\hat{r}$$

On combining these equations we get

$$m_1\frac{d^2\vec{r}_1}{dt^2} + m_2\frac{d^2\vec{r}_2}{dt^2} = 0$$

On integrating we get

$$m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} = const.$$

Or, we can write

$$m_1\vec{v}_1 + m_2\vec{v}_2 = const.$$

$\vec{v}_1, \vec{v}_2$ defined as the velocities of m_1 and m_2.

Therefore the velocity of the centre of mass of the particle is given as

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = const.$$

This shows that the centre of mass moves with a constant velocity.

Self Assessment Question (SAQ) 4: The distance between the centres of carbon and oxygen atoms in the carbon monoxide gas molecule is $1.13\times10^{-10} m$. Calculate the centre of mass of the molecule relative to the carbon atom.

Self Assessment Question (SAQ) 5: Two bodies of mass 10 kg and 2kg are moving with velocities $2\hat{i} - 7\hat{j} + 3\hat{k}$ and $-10\hat{i} + 35\hat{j} - 3\hat{k}$ m/s respectively. Find the velocity of centre of mass.

Self Assessment Question (SAQ) 6: A moving particle of mass m collides head on with the particle of mass 2m which is initially at rest. Show that the particle m will lose $8/9^{th}$ part of its initial kinetic energy after collision.

Self Assessment Question (SAQ) 7: In an inelastic collision of two bodies, what are the quantities which do not change after collision?

Self Assessment Question (SAQ) 8: Is it possible to have a collision in which the whole of kinetic energy is lost?

Self Assessment Question (SAQ) 9: Moon revolves around the earth. Is any work being done on the moon?

Self Assessment Question (SAQ) 10: Two persons A and B lift equally heavy boxes from ground to the same height, but A takes less time in lifting. Has A done more work? Are they equally powerful?

13.7 KEPLER'S LAW

Kepler's laws describe the motion of planets around the Sun. Kepler knew 6 planets: Earth, Venus, Mercury, Mars, Jupiter and Saturn. All these move in nearly the same flat plane. The solar system is flat like a pancake! The Earth is on the pancake, too, so we see the entire system edge-on--the entire pancake occupies one line or maybe a narrow strip cutting across the sky, known as the ecliptic. Every planet, the Moon and Sun too, move along or near the ecliptic. If you see a bunch of bright stars strung out in a line across the sky with the line perhaps also including the Moon, whose orbit is also close to that pancake, or the place on the horizon where the Sun had just set you are probably seeing planets.

Ancient astronomers believed the Earth was the center of the Universe, the stars were on a sphere rotating around it and the planets were moving on their own crystalspheres in funny ways. They usually moved in the same direction, but sometimes their motion reversed for a month or two, and no one knew why.

A Polish clergyman named Nicholas Copernicus figured out by 1543 that those motions made sense if planets moved around the Sun, if the Earth was one of them, and if the more distant ones moved more slowly so sometimes the Earth overtakes them, and they seem to move backwards for a while. The orbits of Venus and Mercury were inside that of Earth, so they never move far from the Sun. Which is why you never see Venus at midnight?

You will probably have heard or read that the pope and church fought the idea of Copernicus, because in one of the psalms, which are really prayer-poems, the bible says that God set up the Earth that it will not move that was one translation: a more correct one is "will not collapse". Galileo, an Italian contemporary of Kepler who supported the ideas of Copernicus, was tried by the church for disobedience and was sentenced to house arrest for the rest of his life.

It was an age when people often followed ancient authors (like the Greek Aristoteles) rather than check out with their own eyes, what Nature was really doing. When people started checking, observing, experimenting and calculating, that became the scientific revolution. Our modern technology is the ultimate result, and Kepler's laws (together with Galileo's work and that of William Gilbert on magnetism) are important, because they started that revolution. There are three laws regarding the motion of planets around the sun.

13.7.1 Kepler's first law

Each planet revolves in an elliptical orbit around the Sun; the Sun is at one of the foci of the ellipse. Sometimes referred to as the law of ellipses - explains that planets are orbiting the sun in a path described as an ellipse. An ellipse can easily be constructed using a pencil, two tacks, a string, a sheet of paper and a piece of cardboard. Tack the sheet of paper to the cardboard using the two tacks. Then tie the string into a loop and wrap the loop around the two tacks. Take your pencil and pull the string until the pencil and two tacks make a triangle, see diagram. Then begin to trace out a path with the pencil, keeping the string wrapped tightly around the tacks. The resulting shape will be an ellipse. An ellipse is a special curve in which the sum of the distances from every point on the curve to two other points is a constant. The two other points (represented here by the tack locations) are known as the foci of the ellipse. The closer together that these points are, the more closely that the ellipse resembles the shape of a circle. In fact, a circle is the special case of an ellipse in which the two foci are at the same location. Kepler's first law is rather simple - all planets orbit the sun in a path that resembles an ellipse, with the sun being located at one of the foci of that ellipse.

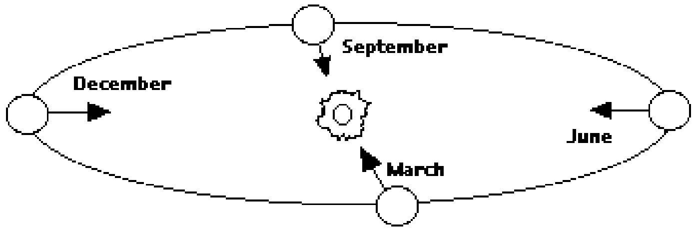

Figure 13.5

13.7.2 Kepler's second law

The radius vector of the planet relative to the Sun sweeps out equal area in equal times that is the areal velocity is constant. Sometimes referred to as the law of equal areas - describes the speed at which any given planet will move while orbiting the sun. The speed at which any planet moves through space is constantly changing. A planet moves fastest when it is closest to the sun and slowest when it is farthest from the Sun. Yet, if an imaginary line were drawn from the center of the planet to the center of the sun, that line would sweep out the same area in equal periods of time. For instance, if an imaginary line were drawn from the earth to the sun, then the area swept out by the line in every 31-day month would be the same. This is shown in the diagram below. As can be observed in the diagram, the areas formed when the earth is closest to the sun can be approximated as a wide but short triangle; whereas the areas formed when the earth is farthest from the sun can be approximated as a narrow but long triangle. These areas are of the same size. Since the base of these triangles are shortest when the earth is farthest from the sun, the earth would have to be moving more slowly in order for this imaginary area to be of the same size as when the earth is closest to the sun.

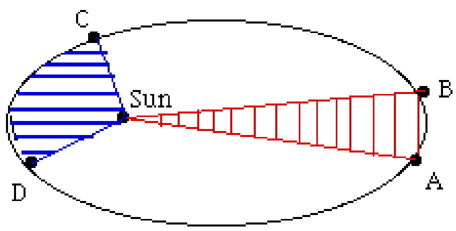

Figure 13.6

13.7.3 Kepler's third law

The square of the period of revolution of any planet around the sun is proportional to the cube of the semi major axis of the elliptical orbit. Sometimes referred to as the law of harmonies - compares the orbital period and radius of orbit of a planet to those of other planets. Unlike Kepler's first and second laws that describe the motion characteristics of a single planet, the third law makes a comparison between the motion characteristics of different planets. The comparison being made is that the ratio of the squares of the periods to the cubes of their average distances from the sun is the same for every one of the planets.

If T is the time period of a planet and r is the distance of planet from the Sun, then

$$T^2 \propto r^3$$

Or, $T^2 = Kr^3$, where K is a constant. From this law it follows that larger is the distance of the planet from the Sun, larger will be the period of revolution. For example, Venus is nearest to Sun, its period is 88 days while the planet Pluto is farthest to Sun, its period is 248 days.

13.7.4 Newton's Conclusion from Kepler's Law

Newton derived important conclusion from Kepler's law. He assumed that the orbit of a planet around the Sun is almost circular.

According to Kepler's second law the areal velocity of a planet around the sun remains constant. Therefore in a circular orbit the linear speed of a planet will remain constant.

From Kepler's third law we have $T^2 \propto r^3$. In a circular orbit a centripetal force acts on a planet, the direction of force is towards centre and its magnitude is given by

$F = \frac{mv^2}{r}$, where m is the mass of the planet, v is its linear speed and r is radius of circular orbit.

Linear speed v= distance travelled in one revolution/ period of revolution

$F = \frac{2\pi r}{T}$, there combining these equations, we get

$$F = \frac{4\pi^2 m}{Kr^2}$$

Or we can write $F \propto \frac{m}{r^2}$, thus Newton's law was derived with following conclusions from Kepler's law:

1. A planet revolving around the Sun is acted upon by force whose direction is towards the sun.
2. The force acting on the planet is directly proportional to the mass of the planet.
3. The force acting on the planet is inversely proportional to the square of the distance of planet from the Sun.

Self Assessment Question (SAQ) 11: Select the correct choice:

1) Which statement is correct
   (a) A central force is always conservative

(b) A central force is always non conservative
(c) Both are correct
(d) None of these correct

2) The total energy of a satellite going around the earth is
(a) Zero
(b) Positive
(c) Negative
(d) Variable

3) The speed of a satellite depends upon
(a) Mass of satellite
(b) Material of satellite
(c) Height of satellite above earth surface
(d) All of above

4) According to Kepler's 2^{nd} law, the radius vector of a planet relative to Sun sweeps out equal areas in equal intervals of time. The law is a consequence of conservation of
(a) Linear momentum
(b) Angular momentum
(c) Energy
(d) Newton's law of gravitation

5) the period of a satellite in a circular orbit of radius R is T. the period of another satellite in a circular orbit of radius 4R is
(a) 4T
(b) t/4
(c) 8T
(d) T/8

13.8 SUMMARY

In this unit, you have studied conservative and non-conservative forces. On which factors these forces depend. You have studied that the central forces are conservative forces. Also, we have derived that force is the negative gradient of potential. You have studied about the collision. We defined there are two types of collision, elastic and non-elastic collision. We also derived that momentum is conserved in elastic collision. You have studied central force, centre of mass coordinates. We have studied in details about the Keplers first laws, second law and third law. The first law states that the shape of each planet's orbit is an ellipse with the sun at one focus. The sun is thus off-center in the ellipse and the planet's distance from the sun varies as the planet moves through one orbit. The second law specifies quantitatively how the speed of a planet increases as its distance from the sun decreases. If an imaginary line is drawn from the sun to the planet, the line will sweep out areas in space that are shaped like pie slices. The second law states that the area swept out in equal periods of time is the same at all points in the orbit. When the planet is far from the sun and moving slowly, the pie slice will be long and narrow; when the planet is near the sun and moving fast, the pie slice will be short and fat. The third law establishes a relation between the average distance of the planet from the sun the semi major axis of the ellipse and the time to complete one revolution around the sun (the period): the ratio of the cube of the semi major axis to the square of the period is the same for all the planets including the earth. Many solved examples are given in the unit to make the concept clear. To check your progress, self assessment questions are given place to place.

13.9 GLOSSARY

Dissipative-wastefully

Intuitively- instinctively

Fundamental- essential

Fright Car- to carry load

Stationary- fixed

Interaction- contact

Ecliptic-apparent path

Constellations- collections

Imaginary- unreal

13.10 TERMINAL QUESTIONS

1. The potential energy of a body is given by

$U = 40 + 6x^2 - 7xy + 8y^2 + 32z$, where U is in jule and x, y, z are in meter. Deduce the components of the force on the body when it is in position (-2, 0, 5).

2. True or False statements
   (i) Conservative forces depend only on the initial and final positions of the body.
   (ii) Conservative force can be expressed as $\vec{F} = -gradU$, where U is the kinetic energy.
   (iii) Gravitational force is an example of non-conservative force.
   (iv) Force of friction is a conservative force.
   (v) Non conservative forces are independent upon path.
   (vi) Curl of conservative force is zero.
3. The maximum and minimum distances of a comet from the Sun are $1.6\times10^{12}\,m$ and $18.0\times10^{10}\,m$ respectively. If the speed of the comet at the nearest point is $6.0\times10^{4}\,m/s$, calculate the speed at the farthest pint.
4. Earth revolution round the Sun has radius $1.5\times10^{11}\,m$ and period $3.15\times10^{7}\,s$. If the gravitational constant is $6.67\times10^{-11}\,Nm^2/Kg^2$, calculate the mass of the Sun.
5. The period of a satellite in circular orbit of radius 12000km around the planet is 3 hours. Obtain the period of a satellite in circular orbit of radius 48000km around the same planet.
6. The mean distance of Mars from Sun is 1.524 times the distance of earth from Sun. Calculate the period of revolution of mars around the Sun.

7. A satellite moves in a circular orbit around the earth at a height R/2 from Earth's surface, where R is the radius of the earth. Calculate its period of revolution. Radius of the earth is $6.38 \times 10^6 m$
8. The period of the Moon is approximately 27.2 days (2.35x106 s). Determine the radius of the Moon's orbit. Mass of the earth = 5.98x1024 kg, T = 2.35x106 s, G = 6.6726 x 10-11N-m2/kg2.
9. Define central force.
10. Give the example of central force.
11. State Kepler's third law.
12. What would happen if a planet suddenly stands still in its orbit?
13. An artificial satellite is resolving around the Earth in an orbit very close to earth. What would happen if the kinetic energy of the satellite is suddenly doubled?
14. What remain constant in the field of central force?
15. Is damping force conservative?

13.11 ANSWERS

Self Assessment Questions (SAQs):

1. $\vec{F} = gradU = -\vec{\nabla}U$

$curl\vec{F} = -\vec{\nabla} \times \vec{F} = -\vec{\nabla} \times (\vec{\nabla}U)$

$= 0$

2. $curl\vec{F} = -\vec{\nabla} \times \vec{F}$

$curl\vec{F} = -\vec{\nabla} \times ((2xy + z^2)\hat{i} + x^2\hat{j} + 2xz\hat{k})$

On solving we get

$curl\vec{F} = 0$

3. $curl\vec{F} = -\vec{\nabla} \times \vec{F}$

$$\vec{F} = ma = m\frac{dv}{dt} = m\frac{d}{dt}\frac{dr}{dt}$$

Also $\theta = \omega t$, using these

$curl\vec{F} = 0$

4. $r_{cm} = \dfrac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$

   $r_{cm} = 0.64 \times 10^{-10} m$, which is along the line of symmetry

5. $\vec{v}_{cm} = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = const.$

$\vec{v}_{cm} = 2\hat{k}m/s$

6. $\dfrac{K_i - K_f}{K_i} = 8/9$
7. In an elastic collision of two bodies the kinetic energy of the system is not conserved. In an elastic collision of two bodies the total linear momentum ant total energy of the system does not change.
8. Yes, in perfectly inelastic collision of two bodies moving towards each other with equal linear momentum.
9. No, the gravitational force acting on moon is always at right angle to the motion of the moon. Hence work done is zero.
10. No, they have done equal work because work does not depend upon time. However, A is more powerful.
11. (a), (c), (c), (b), (c)

Terminal Questions:

1. Given $U = 40 + 6x^2 - 7xy + 8y^2 + 32z$, the x, y, z components of the force at the position (-2, 0, 5) are given by $F_x = -\dfrac{dU}{dx} = -12x + 7y = 24N$, $F_y = -\dfrac{dU}{dy} = 7x - 16y = -14N$

   And $F_z = -\dfrac{dU}{dz} = -32N$

2.
(i) T
(i) F
(ii) F
(iii)F
(iv)F
(v) T

3. by conservation of angular momentum

   $L = mvr = const$

   $mv_1r_1 = mv_2r_2$

   Or, $v = \dfrac{6.0\times10^4 \times 8.0\times10^{10}}{1.6\times10^{12}}$

   $= 3.0\times10^3 m/s$

4. we have $T = \dfrac{4\pi^2 r^3}{GM_s}$

   Therefore,

   $M_s = \dfrac{4\pi^2 r^3}{GT^2}$

$$M_s = \frac{4\times(3.14)^2\times(1.5\times10^{11})^3}{6.67\times10^{-11}\times(3.15\times10^7)^2}$$

$$M_s = 2.0\times10^{30}kg$$

5. the period of a satellite is given by

$$T \propto r^{3/2}$$

If T_1 is the period in an orbit of radius r_1, then

$$\frac{T}{T_1} = (\frac{r}{r_1})^{3/2}$$

$$\frac{3}{T_1} = (\frac{12000}{48000})^{3/2}$$

Or, $T_1 = 24 hours$

6.

$$T \propto r^{3/2}$$

Therefore,

$$\frac{T_{mars}}{T_{earth}} = (\frac{r_{mars}}{r_{earth}})^{3/2}$$

Or,

$$\frac{T_{mars}}{T_{earth}} = (1.524)^{3/2} = 1.88$$

Or, $T_{mars} = T_{earth}\times1.88 = 1.88$ earth year

7. $T = \frac{2\pi(R+h)}{R}\sqrt{\frac{(R+h)}{R}}$

Here, h=R/2

Therefore, $T = \frac{2\pi\times1.5R}{R}\sqrt{\frac{1.5R}{g}}$

$$T = 2\pi\sqrt{\frac{R}{g}}(1.5)^{3/2}$$

$$T = 2\times3.14\sqrt{\frac{6.38\times10^6}{9.8}}(1.5)^{3/2}$$

$$T = 9310s$$

8. We have $M_s = \frac{4\pi^2r^3}{GT^2}$

Or, $r = (\frac{T^2GM}{4\pi^2})^{3/2}$

$$= 3.82\times10^8 m$$

9. In this case only the gravitational force of Sun's attraction would act on the planet and so it would be accelerated towards the sun and fall into the sun.
10. The satellite would escape from the earth.
11. Angular momentum
12. No, it is non conservative.

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UNIT 14 ELASTICITY AND ELASTIC CONSTANT

Structure

14.1 Introduction:

In the course of mechanics unit 6, Block-3 you studied the dynamics of rigid bodies. It means that during the motion of the body if the relative position of constituent particles remains same then the body is termed as **rigid body**. After the better understanding of rigid body you should try to understand the physical concept of non rigid body where the position of the constituent particles in the body changed after the application of external force. This is to be studied in the Block-4 of unit 14 in terms of new terminology ELASTICITY.

In physics, a rigid body is an idealization of a solid body in which deformation is neglected. In other words, the distance between any two given points of a rigid body remains constant in time regardless of external forces exerted on it. Even though such an object cannot physically exist due to relativity, objects can normally be assumed to be perfectly rigid if they are not moving near the speed of light.

Physical properties are properties that can be measured or observed without changing the chemical nature of the substance. All properties of matter are either extensive or intensive and either physical or chemical. Extensive properties, such as mass and volume, depend on the amount of matter that is being measured. Intensive properties, such as density and color, do not depend on the amount of matter. Both extensive and intensive properties are physical properties, which mean they can be measured without changing the substance's chemical identity. For example, the freezing point of a substance is a physical property: when water freezes, it is still water (H2O)—it is just in a different physical state. Some examples of physical properties are:

Color (intensive)

Density (intensive)

Volume (extensive)

Mass (extensive)

Boiling point (intensive): the temperature at which a substance boils

Melting point (intensive): the temperature at which a substance melts

14.2 Objective:

The main objective of this unit is to study in detail about the elasticity and their physical interpretations and also to acquaint the student about the elastic limit, stress, strain, Hooks law, different types of elastic constants and their interrelationships. However, the deformation in the object can occur after the application of external force. The physical interpretation and their concerned relations have been interpreted in their subsequent sections. At last the varities of problems related to different topics have been discussed for better understanding.

14.3 Elasticity:

Elasticity is the ability of a body to resist a distorting influence or stress and to return to its original size and shape when the stress is removed. If the material is elastic, the object will return to its initial shape and size when these forces are removed. The amount of elasticity of a material is determined by two types of material parameter. The first type of material parameter is called a **modulus**, which measures the amount of force per unit area (stress) needed to achieve a given amount of deformation. The units of modulus are Pascal's (Pa).A higher modulus typically indicates that the material is harder to deform. The second type of parameter measures the **elastic limit**. The limit can be a stress beyond which the material no longer behaves elastic and permanent deformation of the material will take place. If the stress is released, the material will elastically return to a permanent deformed shape instead of the original shape.

14.3.1. Strain:

- When a body is under a system of forces or couples in equilibrium then a change is produced in the dimensions of the body.
- This fractional change or deformation produced in the body is called strain.
- Strain is a dimensionless quantity.
- Strain is of three types

(a) Longitudinal strain:- It is defined as the ratio of the change in length to the original length. If L is the original length and ΔL is the change in length then $\Delta L/L$ is termed as longitudinal strain.
Experiments have shown that the change in length (ΔL) depends on only a few variables. As already noted ΔL is proportional to the applied force F and depends on the substance from which the object is made. Additionally, the change in length is proportional to the original length L0 and inversely proportional to the cross-sectional area of the wire or rod. For example, a long guitar string will stretch more than a short one, and a thick string will stretch less than a thin one.

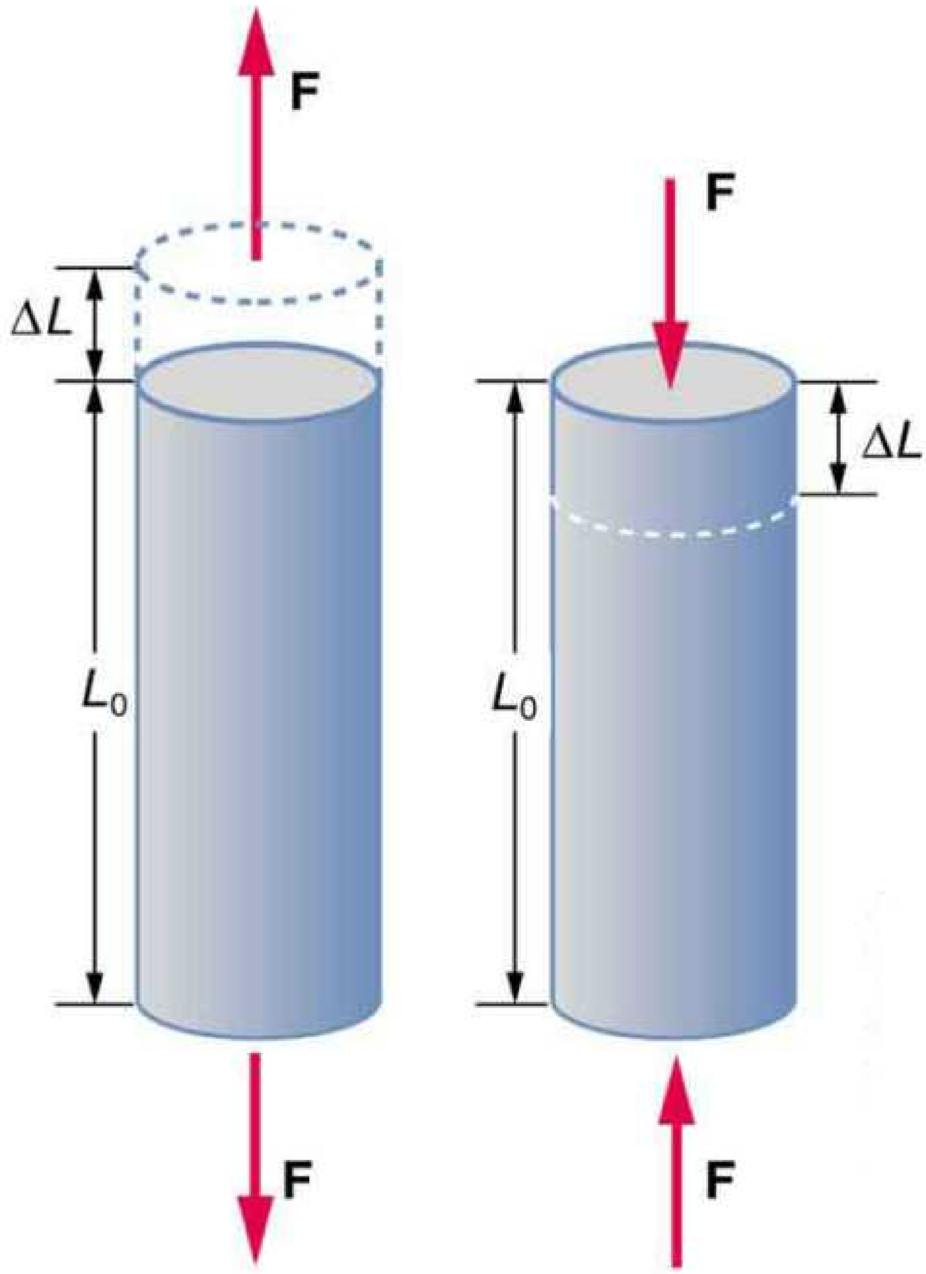


Figure 14.1

(b) Volume strain:-It is defined as the ratio of change in volume to the original volume.

(c) Shearing strain:- If the deforming forces produce change in shape of the body then the strain is called shear strain. In practice since θ is much smaller than l so, tanθ≅ θ and the strain is simply the angle θ(measured in radians). Thus, shear strain is pure number without units as it is ratio of two lengths.

14.3.2 Stress:

When the external deforming forces act on a body, internal forces opposing the former are developed at each section of the body. The magnitude of the internal forces per unit area of the section is called **stress.** In the equilibrium state of the deformed body, the internal forces are equal and opposite of the external forces. Therefore, **Stress is measured by the external forces per unit area of their application.** The dimensions are [$ML^{-1}T^{-2}$] and its units are N/m^2. The details are discussed in the different types of elastic constants. Stress is the force per unit area on a body that tends to cause it to change shape. Stress is a measure of the internal forces in a body between its particles. These internal forces are a reaction to the external forces applied on the body that cause it to separate, compress or slide. External forces are either surface forces or body forces. Stress is the average force per unit area that a particle of a body exerts on an adjacent particle, across an imaginary surface that separates them.

The formula for uniaxial normal stress is:

$$\sigma = \frac{F}{A}.$$

Where σ is the stress, F is the force and A is the surface area.In SI units, force is measured in Newtonand area in square meters.

14.3.3 What is elastic limit:

Elastic limit is the upper limit of deforming force up to which, if deforming force is removed, the body regains its original form completely and beyond which if deforming force is increased, the body loses its property of elasticity and gets permanently deformed. This mobile friendly simulation allows students to stretch and compress spring to explore relationships among force, spring constant, displacement and potential energy in a spring. You can use it to promote understanding of predictable mathematical relationship that underlies Hooke's law. Playing around with this simulation you can get an understanding of **restoring force**.

14.3.4 Stress- Strain curve:

In the curve given below, The part OA is a straight line which shows that up to the point A, stress is directional proportional to strain.i.e. Hooks law is obeyed up to A. The point A is called the limit of proportionality.

If the stress is further increased, a point B known as elastic limit of the material is reached. This point lies near the point A and up to this point; the wire takes back its original length, when the load is removed. Hence for the part AB of the curve, stress is necessarily proportional to strain. On increasing the load beyond elastic limit, the stress-strain curve takes a bend. Now, at any point if the wire is unloaded, it does not regain its original length and gets permanent stretch, which we call **permanent set**. If the wire is now loaded, an entirely new stress-strain curve will represent its behavior.

If the load is further increased, a point C is reached, where the strain is much greater for a small increase in the load. This point C is called the **Yield Point** and the corresponding stress being the

yield stress. Beyond this point C, the extension increases rapidly without an increase in the load, i.e. the material of the wire flows beyond C. This is known as **plastic flow**. As the wire becomes thin, the stress becomes considerably greater and the wire cannot support the same as before and wire is to be prevented from being broken, the load must be diminished.
After crossing the yield point, the thinning of wire no longer remains uniform and the diameter of a section decreases considerably. Now, the wire shows a phenomenon, known as 'necking'. Immediately, as this occur, the stress decreases automatically and the portion EF of the curve is obtained; ultimately a point F is known as **Breaking- point**, is reached, at which the wire breaks. The stress corresponding to this point is F is known as **breaking stress**. Here, the stress corresponding to E is called the **ultimate strength** or **tensile strength** of the given material. The tensile strength of the material is defined as the ratio of maximum load to which the specimen wire may be subjected by slowly increasing load to the original cross-sectional area of the wire. The ultimate strength of a material is, however measured by the stress causing the test specimen to break.

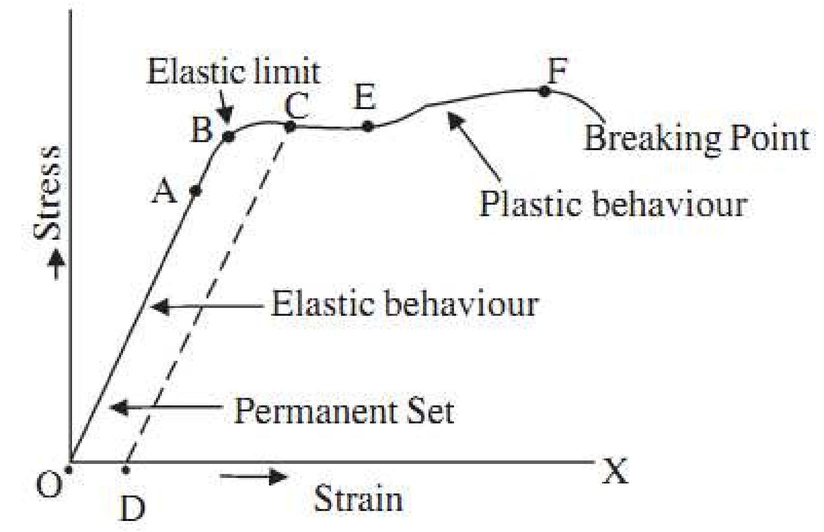


Figure 14.2

14.4 Hooke's Law:

Robert Hooke (1635-1703) invented the law as, within the proportional limit stress is directly proportional to strain. When studying springs and elasticity, the 17th century physicist Robert *Hooke* noticed that the stress vs strain curve for many materials has a linear region. Within certain limits, the force required to stretch an elastic object such as a metal spring is directly proportional to the extension of the spring.

- Hook's law is the fundamental law of elasticity and is stated as "for small deformations stress is proportional to strain".

Thus,
$stress \propto strain$
or,
stress /strain = constant

- This constant is known as modulus of elasticity of a given material, which depends upon the nature of the material of the body and the manner in which body is deformed.
- Hook's law is not valid for plastic materials.
- Units and dimension of the modulus of elasticity are same as those of stress.

Example for Understanding:

The spring is a marvel of human engineering and creativity. For one, it comes in so many varieties – the compression spring, the extension spring, the torsion spring, the coil spring, etc. – all of which serve different and specific functions. These functions in turn allow for the creation of many man-made objects, most of which emerged as part of the Scientific Revolution during the late 17th and 18th centuries.

As an elastic object used to store mechanical energy, the applications for them are extensive, making possible such things as an automotive suspension systems, pendulum clocks, hand sheers, wind-up toys, watches, rat traps, digital micromirror devices, and of course, the slinky. Like so many other devices invented over the centuries, a basic understanding of the mechanics is required before it can so widely used.

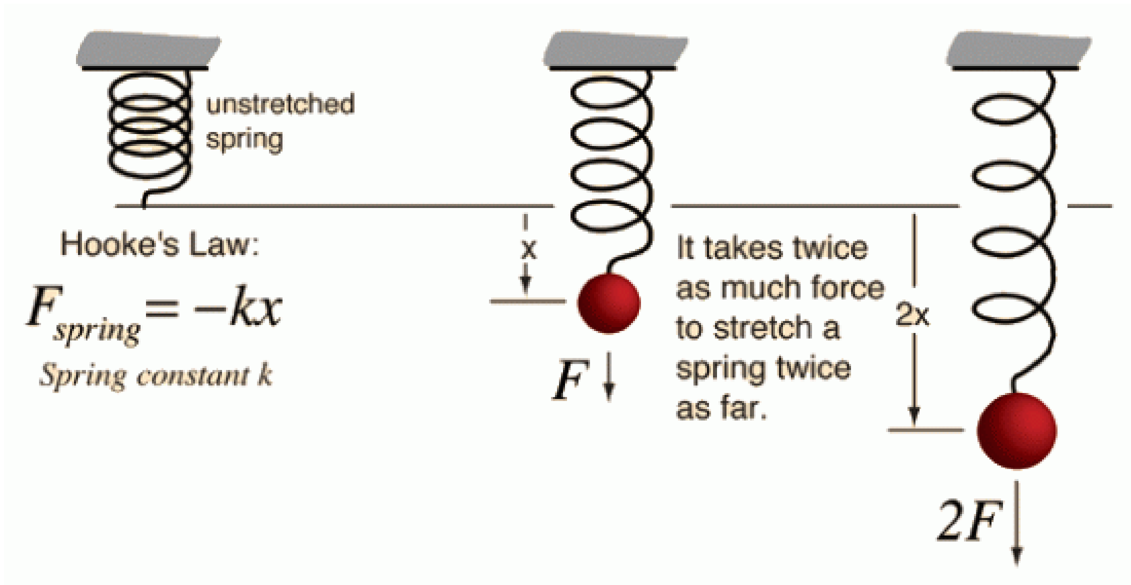


Figure 14.3: Illustration of Hooke's Law, showing the relationship between force and distance when applied to a spring.

This can be expressed mathematically as $F = -kX$, where F is the force applied to the spring (either in the form of strain or stress); X is the displacement of the spring, with a negative value demonstrating that the displacement of the spring once it is stretched; and *k* is the spring constant.

Hooke's law is the first classical example of an explanation of elasticity – which is the property of an object or material which causes it to be restored to its original shape after distortion. This ability to return to a normal shape after experiencing distortion can be referred to as a "restoring force". Understood in terms of Hooke's Law, this restoring force is generally proportional to the amount of "stretch" experienced.

In addition to governing the behavior of springs, Hooke's Law also applies in many other situations where an elastic body is deformed. These can include anything from inflating a balloon and pulling on a rubber band to measuring the amount of wind force needed to make a tall building bend and sway.

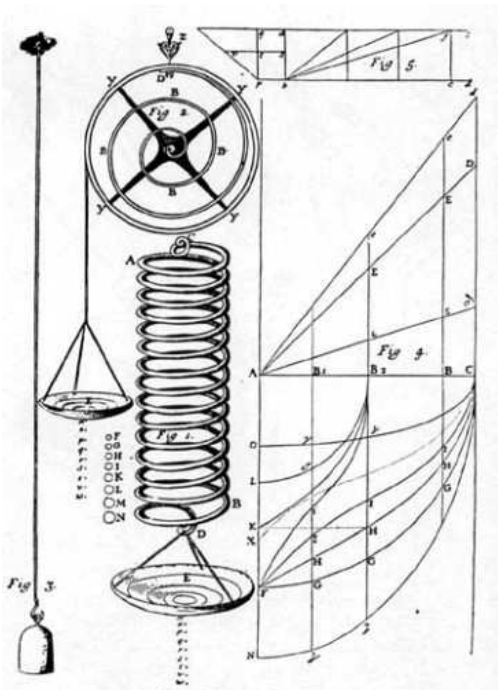


PLATE TO HOOKE'S LECTURE 'OF SPRING' 1678.

FIG. 1. Wire helical spring stretched to points *r, p, q, r, s, t, v, w*, by weights F, G, H, I, K, L, M, N.
FIG. 2. Watch spring similarly stretched by weights put in pan.
FIG. 3. The 'Springing of a string of Brass Wire 36 ft. long'.
FIG. 4. Diagram of velocities of springs.
FIG. 5. Diagram of law of ascent and descent of heavy bodies.

Figure 14.4 Illustration from Hooke's 1678 treaties "De potentiarestitutiva (Of Spring)" Source: umn.edu

This law has had many important practical applications, with one being the creation of a balance wheel, which made possible the creation of the mechanical clock, the portable timepiece, the spring scale and the manometer (the pressure gauge). Also, because it is a close approximation of all solid bodies (as long as the forces of deformation are small enough), numerous branches of science and engineering as also indebted to Hooke for coming up with this law. These include the disciplines of seismology, molecular mechanics and acoustics.

However, like most of the classical mechanics, Hooke's Law only works within a limited frame of reference. Because no material can be compressed beyond a certain minimum size (or stretched beyond a maximum size) without some permanent deformation or change of state, it only applies so long as a limited amount of force or deformation is involved. In fact, many materials will noticeably deviate from Hooke's law well before those elastic limits are reached.

Still, in its general form, Hooke's Law is compatible with Newton's laws of static equilibrium. Together, they make it possible to deduce the relationship between strain and stress for complex objects in terms of the intrinsic material properties. For example, one can deduce that a homogeneous rod with uniform cross section will behave like a simple spring when stretched, with a stiffness (*k*) directly proportional to its cross-section area and inversely proportional to its length.

14.5 Elastic Constants:

14.5.1Young's Modulus of Elasticity:

- Young's Modulus of elasticity is the ratio of longitudinal stress to longitudinal strain, within elastic limit.
- It is denoted by Y.
- Young's Modulus of elasticity is given by

$$Y = \frac{longitudinal\ stress}{lingitudinal\ strain}$$

- Let us now consider a wire of length l having area of cross-section equal to A. If the force F acting on the wire, stretches the wire by length Δ l then

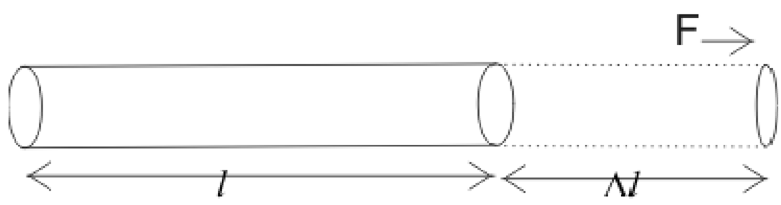


Figure 14.4

$$longitudinal\ stress = \frac{F}{A} \qquad (1)$$

and

$$lingitudinal\ strain = \frac{\Delta l}{l} \qquad (2)$$

From (1) and (2) we have Young's modulus of elasticity as

$$Y = \frac{Fl}{A\Delta l}$$

- Young's modulus of elasticity has dimensions of force/Area i.e. of pressure.
- Unit of Young's modulus is N/m^2.
- If area of cross-section of a wire is given by $A = \pi r^2$ then Young's modulus is

$$Y = \frac{Fl}{\pi r^2 \Delta l}$$

again if $A = \pi r^2 = 1cm^2$ and $\Delta l = l = 1cm$ then
Y = F
Thus, ***Young's modulus can also be defined as the force required to double the length of a wire of unit length and unit area of cross-section.***

14.5.2 Bulk Modulus of Elasticity:

- The ratio of normal stress to volume strain within elastic limits is called Bulk Modulus of elasticity of a given material.
- It is denoted by K.
- Suppose a force F is applied normal to a surface of a body having cross-sectional area equal to A, so as to cause a change in it's volume.
  If applied force bring about a change ΔV in the volume of the body and V is the original volume of the body then,

$$normal\ stress = \frac{F}{A}$$

and

Volume strain = ΔV/V

So, Bulk Modulus of elasticity would be,

$$Bulk\ modulus = K = \frac{normal\ stress}{volume\ strain}$$

Thus,

$$K = \frac{FV}{A\Delta V}$$

- For gases and liquids the normal stress is caused by change in pressure i. e., normal stress = change in pressure ΔP.
  Thus, bulk Modulus is

$$K = -\frac{V\Delta P}{\Delta V}$$

here negative sign indicates that the volume decreases if pressure increases and vice-versa.

- For extremely small changes in pressure and volume, the Bulk Modulus is given by

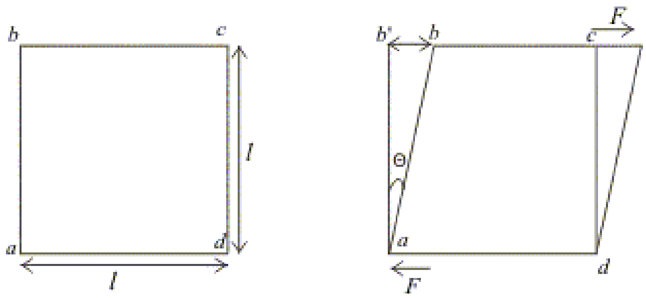
$$K = -V\frac{dP}{dV}$$

- Reciprocal of Bulk Modulus is called compressibility of the substance. Thus,

$$compressibility = \frac{1}{K} = \frac{\Delta V}{V\Delta P}$$

14.5.3 Modulus of Rigidity (η):

- When a body is sheared, the ratio of tangential stress to the shearing strain within elastic limits is called the Modulus of Rigidity and it is denoted by η.
- If lower face of the rectangular block shown below in the figure, is fixed and tangential force is applied at the upper face of area A, then shape of rectangular block changes.



A block subjected to shearing stress deforms by an angle θ (Front face shown)

Figure 14.6

So,

shearing strain = $\theta \cong \tan\theta = \frac{bb'}{l} = \frac{x}{l}$ (where bb' = x, displacement of upper face)

or,

$$tengential\ stress = \frac{F}{A}$$

Thus,

$$modulus\ of\ rigidity = \eta = \frac{Fl}{xA}$$

14.5.4 Poisson's Ratio:

The ratio of lateral strain to linear strain is called Poisson's ratio. It is denoted by 'σ'. The lateral strain is defined as the ratio of change in diameter to original diameter. If a wire of length L and diameter D, is elongated by pulling to length (L+ l), it's literal dimension (diameter) decreases to (D-d).

σ = lateral strain/linear strain.

$$= \frac{(d/D)}{(l/L)}$$

The value of 'σ' varies from 1/3 to 1/4 depending upon the material. If τ is the applied tensile stress and γ is the young's modulus of the wire, then linear strain is (τ/γ) and lateral strain is $\sigma.(\tau/\gamma)$

14.5.5 Points you must note about elastic modulus:

1. The value of elastic modulus is independent of stress and strain. It depends only on the nature of the material.
2. Greater value of modulus of elasticity means that the material has more elasticity i.e., material is more elastic.
3. Young's Modulus and Shear Modulus exists only for solids while Bulk Modulus is defined for all three stats of matter.
4. Three modulus of elasticity Y, η and K depends on temperature. Their value decreases with the increase in temperature.
5. In case of longitudinal stress, shape remains unchanged while the volume changes. In tensile one volume increases while in compressive one volume decreases.
6. In shear stress, volume remains the same but shape changes.
7. In volume stress, volume changes but shape remains the same.

14.6 Relation between elastic constants

14.6.1 Relation between Y, η and σ:

$$Y = 2\eta\,[1+\sigma]$$

14.6.2 Relation between Y, σ and K:

Y = 3K (1-2σ)

14.6.3 Relation between Y, η and K:

Y=9Kη/ (3K+η)

14.7 DERIVATION OF RELATION AMONG ELASTIC CONSTANTS:

14.7.1 Derivation for the Relation between Y, η and σ:

Let us establish a relation among the elastic constants Y, η and σ. Consider a cube of material of side 'a' subjected to the action of the shear and complementary shear stresses as shown in the figure and producing the strained shape as shown in the figure below. Assuming that the strains are small and the angle A C B may be taken as 45^0.

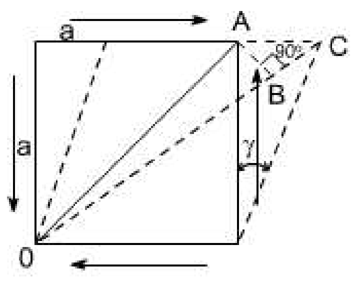


Figure 14.7

Therefore, strain on the diagonal OA = Change in length / original length

Since angle between OA and OB is very small hence OA = OB therefore BC, is the change in the length of the diagonal OA

Thus, strain on diagonal OA = BC/OA

$= AC \cos 45^0 / OA$

$OA = a/ \sin 45^0$

$= a. \sqrt{2}$

Hence, tensile strain = AC/a$\sqrt{2}$. 1/ $\sqrt{2}$

γ=AC / 2a

but AC = a γ (as tan $\gamma \cong \gamma$ = AC/a) , where γ = shear strain

Thus, the strain on diagonal = ar /2a = r / 2

From the definition, If τ is the applied sharing stress, then

$\eta = \tau/\gamma$ or

$\gamma = \tau/\eta$

Thus, the strain on the diagonal= γ /2 = τ / 2η.

Now this shear stress system is equivalent or can be replaced by a system of direct stresses at 45^0 as shown below. One set will be compressive, the other tensile, and both will be equal in value to the applied shear strain stress τ, thus decreasing diagonal DN and increasing OC in length.

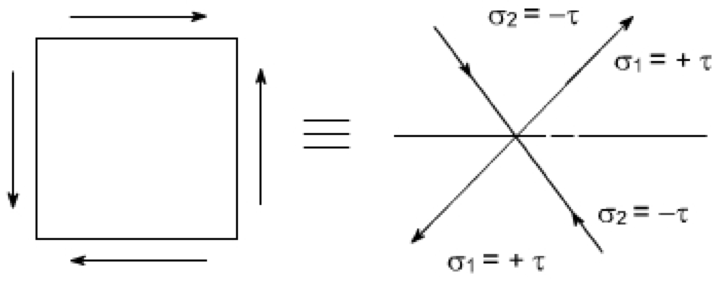


Figure 14.8

Thus, for the direct state of stress system which applies along the diagonals:

strain on diagonal = σ_1 = σ_1/Y - σ. σ_2 /Y, (a compressive strain τ_2/γ is equivalent to a tensile strain $\tau \times (\tau_2/\gamma)$ in the lateral direction OA, increasing it's length)

=τ/ Y- σ. (-τ)/Y

= τ/ Y(1+σ).

Equating the two strains one may get

$$\tau/2\eta = \tau(1+\sigma)/ Y$$

$$Y= 2\eta(1+\sigma).$$

We have introduced a total of four elastic constants, i.e Y, η, K and σ. It turns out that not all of these are independent of the others. In fact given any two of them, the other two can be found.

Again $$Y= 3K(1-2\sigma)$$

$$K = Y/\{3(1-2\sigma)\}$$

When σ = 0.5, the value of E is infinite, rather than a zero value of E and volumetric strain is zero, or in other words, the material is incompressible.

14.7.2 Derivation for the Relation between Y, K and σ:

Consider a cube subjected to three equal stresses τ as shown in the figure below, due to which it is being expanded in all directions.

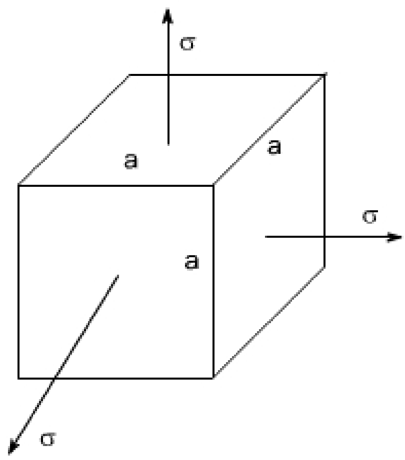


Figure 14.9

The total strain in one direction or along one edge due to the application of hydrostatic stress or volumetric stress τ is given as;

Liner strain along one edge =

$\tau/Y - \sigma.\tau/Y - \sigma.\tau/Y$

$\tau\ (1- 2\sigma) / Y$

volumetric strain = 3 times of linear strain

$$= 3.\tau\ (1- 2\sigma)/ Y.$$

By definition

Bulk modulus of elasticity (K) = Volumetric stress / Volumetric strain

or

Volumetric strain = τ / K

Equating the two strains we get

$$Y= 3K(1-2\sigma)$$

14.7.3 Derivation for the Relation between Y, η and K:

The relationship between Y, η and K can be easily determined by eliminating σ from the already derived relations

$$Y = 2 \eta (1 + \sigma) \text{ and } Y = 3 K (1 -2 \sigma).$$

Thus, the following relationship may be obtained

$$Y = 9K\eta/ (3K+\eta).$$

14.7.4 Derivation for the Relation between η, K and σ:

From the already derived relations, Y can be eliminated

$$Y = 2\eta [1+ \sigma]$$

$$Y = 3K (1-2 \sigma)$$

Thus, we get

$$2\eta [1+ \sigma] = 3K (1-2 \sigma)$$

$$\sigma= 0.5(3K-2\eta)(\eta+3K).$$

14.8 Summary:

In this unit, you have studied about elastic materials and their elastic properties. To present the clear understanding of elasticity and different elastic constants, the elastic limit, Hooke's law and enter relationships between elastic constants have been discussed. The derivations of elastic constants in a different way are given in this chapter. Applications of elasticity in the field of science and technology have been described. The pictorial understanding of elastic constants such as Young modulus Y, Bulk modulus K and Modulus of rigidity η have been discussed.

14.9 Glossary:

Elastic – Regain – return into original shape

Elastic limit – a region to follow Hooke's law, stress proportional to strain

Stress – force per unit area – pressure

Limit- within the defined range

Shear – some deformation from original one

Confined- restricted

Undergo- suffer

Maintain- sustain

Resist- refuse to go along with

Strain – ratio of change in dimension to original dimension

Compressibility – reciprocal of bulk modulus

14.10 Terminal Questions:

14.10.1 Multiple Choice Questions:

1. Maximum limit up to which stress is applied on body without deformation is called

   a. limit
   b. elastic limit
   c. strain
   d. none of above

2. If a 1 m long steel wire having area 5 x 10^{-5} is stretched through 1 mm by force of 10, 000 N then young modulus of wire is
   a. 2 x 10^{11} N m^{-2}
   b. 3 N m^{-4}
   c. 4 N m^{-2}
   d. 5 N m^{-2}

3. Ratio of stress to strain is
   a. 1

b. 0
c. 3
d. constant

4. Stress is
    a. External force
    b. Internal resistive force
    c. Axial force
    d. Radial force

5. Which of the following is not a basic type of strain?
    a. Compressive strain
    b. Shear strain
    c. Area strain
    d. Volume strain

6. Tensile Strain is
    a. Increase in length / original length
    b.Decrease in length / original length
    c. Change in volume / original volume
    d. All of the above

7. Compressive Strain is
    a. Increase in length / original length
    b. Decrease in length / original length
    c. Change in volume / original volume
    d. All of the above

8. Hooke's law is applicable within
    a. Elastic limit
    b. Plastic limit
    c. Fracture point
    d. Ultimate strength

9. Young's Modulus of elasticity is
    a. Tensile stress / Tensile strain
    b. Shear stress / Shear strain
    c. Tensile stress / Shear strain
    d. Shear stress / Tensile strain

10. Maximum limit up to which stress is applied on body without deformation is called
    a. limit
    b. elastic limit
    c. strain

d. none of above.

(Ans: 1-b, 2-a, 3-d, 4-b, 5-c, 6-a, 7-b, 8-a, 9-a, 10-b)

1. If a 1 m long steel wire having area 5 * 10^{-5} is stretched through 1 mm by force of 10, 000 N then young modulus of wire is

a. 2X 10^{11} N m^{-2}

b. 3 N m^{-4}

c. 4 N m^{-2}

d. 5 N m^{-2}

2. Ratio of stress to strain is

a. 1

b. 0

c. 3

d. constant

3. Which of the following material is more elastic?

(a) Rubber

(b) Glass

(c) Steel

(d) Wood

4. A load of 1 kN acts on a bar having cross-sectional area 0.8 cm^2 and length 10 cm. The stress developed in the bar is

(a) 12.5 N/mm^2

(b) 25 N/mm^2

(c) 50 N/mm^2

(d) 75 N/mm^2

5. A brittle material has

(a) No elastic zone

(b) No plastic zone

(c) Large plastic zone

(d) None of these

6. The length of a wire is increased by 1 mm on the application of a certain load. In a wire of the same material but of twice the length and half the radius, the same force will produce an elongation of

(a) 0.5 mm

(b) 2 mm

(c) 4 mm

(d) 8 mm.

14.10.2 Solved Problems:

Question 1. A block of gelatin is 60 mm by 60 mm by 20 mm when unstressed. A force of .245 N is applied tangentially to the upper surface causing a 5 mm displacement relative to the lower surface. The block is placed such that 60X60 comes on the lower and upper surface. Find the shearing stress, shearing strain and shear modulus

(a) (68.1 N/m^2 , .25 , 272.4 N/m^2)
(b) (68 N/m^2 , .25 , 272 N/m^2)
(c) (67 N/m^2 , .26 , 270.4 N/m^2)
(d) (68.5 N/m^2 , .27 , 272.4 N/m^2)

Solution:
Shear stress=F/A=$.24536\times10^{-4}$=68.1N/m^2

Shear strain=tanθ=d/h=5/20=.25

Shear modulus

strain=272.4N/m

Question 2.A steel wire of diameter 4 mm has a breaking strength of $4X10^5$N. The breaking strength of similar steel wire of diameter 2 mm is

(a)$1X10^5$N.
(b) $4X10^5$N.
(c) $16X10^5$N.
(d) none of the these

Solution
Breaking strength is proportional to square of diameter,Since diameter becomes half,Breaking strength reduced by 1/4. Hence A is correct.

Question 3.What is the SI unit of modulus of elasticity of a substance?

(a) Nm^{-1}
(b) Nm^{-2}
(c) Jm^{-1}
(d) Unit less quantity

Solution
Answer is b

Question 4A thick uniform rubber rope of density 1.5 gcm^{-3} and Young Modulus $5X1010^{6}$ Nm^{-2} has a length 8 m. when hung from the ceiling of the room, the increase in length due to its own weight would be ?

(a) .86m
(b) .2m
(c) .1m
(d) .096m

Solution The weight of the rope can be assumed to act at its mid point. Now the extension x is proportional to the original length L. if the weight of the rope acts at its midpoint, the extension will be that produced by the half of the rope. So replacing L by L2L2 in the expression for Young 's Modulus. Substituting the values, we get
l=.096ml=.096m

14.11 Numerical Questions:

Problem 1: A spring stretches 5 cm when a load of 20 N is hung on it. If instead, we put a load of 30 N, how much will the spring stretch? What is the spring constant?

Solution: There are a couple of ways to solve this problem.
Way #1: Notice that 30 N = 20 N + 10 N
20 N creates a stretch of 5 cm. Since 10 N is half of 20 N, then 10 N will create a stretch that is half of 5 cm or 2.5 cm.
Total stretch = 5 cm + 2.5 cm = 7.5 cm
Way #2: Set up a proportion.
5 cm is to 20 N as 'new stretch' is to 30 N.
5 × 30 = new stretch × 20
150 = new stretch × 20
new stretch = 7.5 cm
To get the spring constant, make a couple of good observation.
20 = 4 × 5
30 = 4 × 7.5
F = 4 × x
F is the force applied and x is the stretch
The spring constant is k = 4

Problem 2: With a weight of 25 kg, a spring stretches 6 cm. Its elastic limit is reached with a weight of 150 kg. How far did the spring stretch?
Since 150 kg divided by 25 kg = 6 kg, 150 kg is 6 times bigger.
The stretch will then be 6 times bigger than 6 cm or 36 cm.

Problem 3: A spring has a spring constant that is equal to 3.5. What force (in kilograms) will make it stretch 4 cm?
$F = k \times x$
$F = 3.5 \times 4$
$F = 14$ kg

Problem 4: When the weight hung on a spring is increased by 60 N, the new stretch is 15 cm more. If the original stretch is 5 cm, what is the original weight?
We will need some algebra and a proportion to solve this tough word problem.
Let x be the original weight, then x + 60 is the new weight
If the original stretch is 5 cm, then the new stretch is 20 cm.

$$x \times 20 = 5 \times x + 5 \times 60$$

$$20x = 5x + 300$$

$$15x = 300$$

Since $15 \times 20 = 300$, the original weight is 20 N.

14.12 References:

1. Mechanism by Dr. L. P. Verma, Published from S. J. Publication, P. Kumar and G. L. Lohani.
2. Properties of matter by J. C. Upadhyay, KedarNath Publication.
3. Mechanism by D. S. Mathur, S. Chand & Company.
4. Mechanism by C. L. Arora- New age international.
5. Mechanism by Suresh Chandra- Narosa Publication company, New Delhi.
6. Mechanism by R. B. Singh & D. N. TripathiByKedarNath.
7.Elementary Mechanics, IGNOU, New Delhi
8. 9. Objective Physics, SatyaPrakash, AS Prakashan, Meerut
10. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker, John Wiley & Sons
11. Concepts of Physics, Part I, HC Verma, BharatiBhawan, Patna

14.13 Suggested Readings:

1. Properties of matter by J. C. Upadhyay, KedarNath Publication.

2. Mechanism by Suresh Chandra- Narosa Publication company, New Delhi.

3. Physics Part-I, Robert Resnick and David Halliday, Wiley Eastern Ltd

4. Berkeley Physics Course Vol I, Mechanics, C Kittel et al, McGraw- Hill Company

UNIT 15: TORSION OF CYLINDER AND BENDING OF BEAM

Structure

15.1 Objectives:

The unit introduces the cases of a rigid body where twisting couple is acting on it. The theory of bending is also developed with reference to circular as well as rectangular beams. After going through this unit the student will be able-

(i) To find the twisting couple acting on a cylindrical rod.
(ii) To find the bending moment and depression of a beam under the applied load.
(iii) To know why the shape of girder is I.

15.2 Introduction:

In the previous units we have studied that every material gets deformed to a greater or smaller extent, depending upon the way in which the stress acts.

So, whenever we have to design any structure, we have to give due consideration to the possible stress that the material may suffer at one or the other stage. For example, in building of bridges, railway tracks, ceilings of buildings, etc. different types of beams are used. The material and their characteristics are discussed here.

15.3 Torsion of cylinder:

15.3.1 Angle of twist:

Let us consider a cylindrical rod or wire of length 'l' and radius 'r' clamped at the upper end and twisted by a couple at the lower free end (fig. 15.1). By doing so each circular cross-section of the rod rotates through an angle. This angle is called the angle of twist and is proportional to the distance of cross-section from the clamped end. If the angle of twist at the free end is θ, then clearly

$BB' = r\,\theta \qquad (1)$

where θ is the angle of twist.

15.3.2 Angle of shear:

From fig. 15.1

$BB' = l\,\Phi \qquad (2)$

This Φ is called angle of shear. Thus it is clear that

BB’= r θ= l Φ   (3)

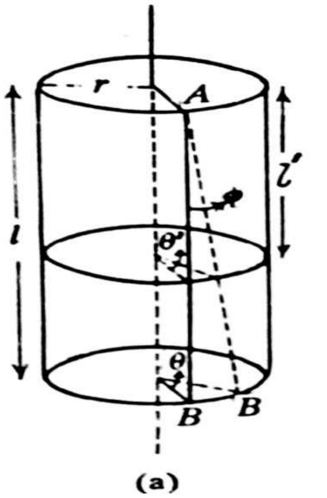


Figure.15.1

The figure also makes it clear that Φ is constant for a cylindrical surface of radius ‘r’. In any section normal to the axis at a distance, say ‘l’ from the fixed end, let the angle of twist be θ’, then

r θ’=l ’Φ   (4)

Comparing (3) and (4), we see that l’< l, hence θ’< θ.

Thus we find that the angle of twist, which is constant in any section normal to the axis at a given length from the fixed end, goes on decreasing as the distance from the fixed end is decreased. At the fixed end, l=0, hence θ=0.

Fig. 15.2 shows the shearing of any two concentric cylindrical layers of radius r and r’. The angle of twist θ at the lower free end is a constant. For the outer cylindrical surface

l Φ= r θ

and for the inner cylindrical layer

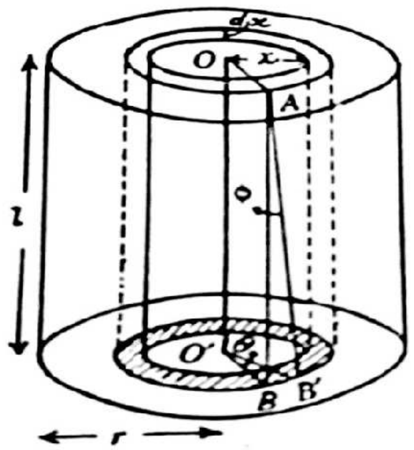


Figure 15.2

$l\,\Phi' = r'\,\theta$

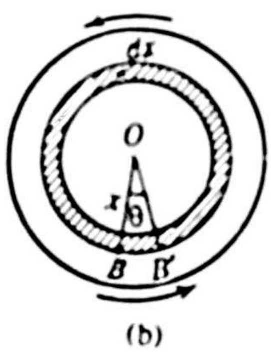
as $r' < r$,

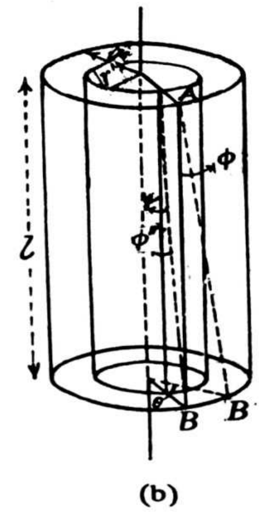
therefore, $\Phi' < \Phi$

Hence, we see that the angle of shear, a constant for any cylindrical surface, goes on decreasing as the radius of the surface decreases and that for r=0, Φ=0, it is maximum on the surface of the cylinder.

15.3.3 Twisting couple on a cylinder:

(i) Case of solid cylinder





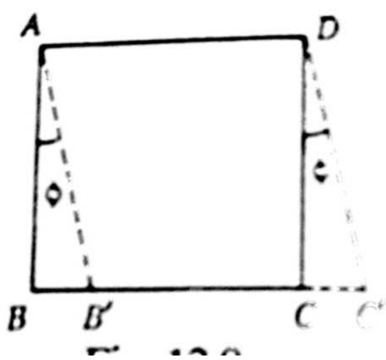


Fig15.3(a) Fig 15.3(b) Fig15.3(c)

We have already discussed that if a couple is applied to the lower end of a cylinder of length L and radius R in a plane perpendicular to its length, then it is twisted through an angle θ. Due to the elasticity of the material of the rod, a restoring couple is set up in it to resist or oppose the twisting couple. In the position of equilibrium, the two couples balance each other.

To obtain the value of this couple, let us imagine the cylinder to consist of a large number of hollow, coaxial cylinder, one inside the other and consider one such cylinder of radius x and thickness dx (fig. 15.3(a)). It's quite clear that each radius of the base of the cylinder will turn through the same angle θ but the displacement (BB') will be maximum at the rim, progressively decreasing to zero at the centre O (fig. 15.3(a) and (b)) indicating that stress is not uniform all over.

Thus, a straight line AB, initially parallel to the axis OO' of the cylinder will take up the position AB' or the angle of shear, ∠BAB'=Φ. This can also be visualised if we imagine the hollow cylinder to be cut along AB and spread out when it will initially have a rectangular shape ABCD (fig. 5.3(c)) and will acquire the shape of a parallelogram AB'C'D after it has been twisted, so that the angle of shear= ∠BAB'=Φ.

Now, BB'= xθ= LΦ, hence, the shear, Φ= xθ/L and will obviously have the maximum value when x=R, i.e. at the outermost part of the cylinder and least at the innermost.

If F be the tangential force acting on the base of this thin cylindrical shell, producing a shear, then

$$Tangential\ stress = \frac{F}{area\ of\ the\ base\ of\ the\ shell}$$

But, area of the base of the shell= circumference * thickness

$$= 2\pi x\ dx$$

$\therefore\ \ tangential\ stress = \frac{F}{2\pi x dx}$

If η is the modulus of rigidity of the material of rod, then

$$\eta = \frac{Tangential\ stress}{shear} = \frac{F}{2\pi x dx.\Phi}$$

$$\eta = \frac{F}{2\pi x dx}.\frac{L}{x\theta}$$

Hence, $F = \frac{2\pi\eta\theta}{L}$ x^2dx

The moment of this force about the axis OO' of the cylinder

$$= \frac{2\pi\eta\theta}{L} x^2 dx.x$$

$$= \frac{2\pi\eta\theta}{L} x^3 \, dx$$

∴ Twisting couple on the whole cylinder

$$= \int_0^R \frac{2\pi\eta\theta}{L} x^3 \, dx$$

$$= \frac{2\pi\eta\theta}{L} \text{ x } \frac{R^4}{4}$$

$$= \frac{\pi\eta R^4 \, \theta}{2L}$$

If θ=1, then twisting couple per unit twist= $\frac{\eta\pi R^4}{2L}$ = C .

C is also called torsional rigidity.

(ii) Case of hollow cylinder

If the cylinder is hollow, one of inner and outer radius R_1 and R_2 respectively, then we have twisting couple on the cylinder

$$= \int_{R1}^{R2} \frac{2\pi\eta\theta}{L} x^2 \, dx$$

$$= \frac{\pi\eta}{2L} [R_2^4 - R_1^4]\theta$$

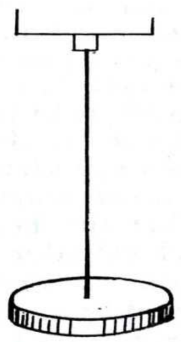
It can be shown that c'>c, i.e. the twisting couple per unit twist is greater for a hollow cylinder than for a solid one of the same material, mass and length.

This explains the use of hollow shafts, in preference to solid ones, for transmitting large torques in a rotator machinery.

15.3.4 Torsional pendulum:

If one end of a fairly thin and long wire is clamped to a rigid support and the other end is attached to the centre of a heavy body like a disc or a sphere, then this arrangement is called the torsional pendulum.

If the disc or sphere be turned in the horizontal plane to twist the wire and then released, it executes torsional vibrations of a definite period about the wire as axis. Let Φ be the angle at any instant, through which the body is twisted. Then the restoring couple set up in the wire is given by



-cΦ

Fig.15.4

where c is the restoring couple per unit twist. The negative sign is used because the couple is directed opposite to the twist Φ. Now if I is the moment of inertia of the disc about the axis of oscillation, the couple acting upon it at this instant must be Iα, where α is the angular acceleration at this instant.

$$I\alpha = -c\Phi$$

$$I\frac{d^2\Phi}{dt^2} = -c\Phi$$

$$\frac{d^2\Phi}{dt^2} = \frac{-c}{I}\Phi$$

$$= -\omega^2\Phi$$

$$\omega^2 = \frac{c}{I} = constant$$

or

$$\frac{d^2\Phi}{dt^2}\,\alpha - \Phi$$

Thus the angular acceleration is proportional to the twist or the angular displacement. Hence the motion is simple harmonic, and its time period is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{C}}$$

Determination of Modulus of Rigidity :-

If the disc attached to the lower end of wire is given slight rotation by hand towards left or right, the system begins to perform torsional oscillations. Period of this oscillation is given by

$$T = 2\pi\sqrt{\frac{I}{C}}$$

where I is the moment of inertia of the disc about the axis of rotation and C is the torsional constant of the wire.

$$T = 2\pi\sqrt{\frac{I\ 2L}{\eta\pi r^4}}$$

The value of I, for a disc about the wire is $I = MR^2/2$, hence

$$T = 2\pi\sqrt{\frac{MR^2}{2}\ \frac{2L}{\eta\pi r^4}}$$

$$T^2 = 4\pi^2 \times \frac{MR^2}{\eta\pi r^4} L$$

$$\eta = \frac{4\pi^2\ MR^2\ L}{T^2\pi\ r^4}$$

$$= \frac{4\pi MR^2\ L}{T^2\ r^4}$$

The time period T is measured with the help of stop watch. The radius r of the wire is measured by a screw gauge at several places of the wire in two perpendicular directions and mean value of 'r' is calculated. The length L of the wire is measured by a metre scale. Thus, by substituting all the values in above formula, the modulus of rigidity η is calculated.

There are two drawbacks of torsional pendulum :

(i) While deriving the above formula for η, we have assumed c to be constant, which is not correct. With change in load at the end of wire, the radius of wire also changes.

(ii) We take the density of body to be constant throughout while calculating moment of inertia, which is not so in practise.

Both the errors are removed in Maxwell's needle method.

15.3.5 Maxwell's needle:

Maxwell's needle is a modification of ordinary torsional pendulum. Maxwell's needle consists of a long hollow brass tube of length L, suspended horizontally by a vertical wire, whose modulus of rigidity is to be determined. The other end of the wire is connected to a rigid support. Four brass cylinders, two hollow and two solid, of identical lengths and diameters, can be fitted inside the hollow brass tube. The length of each cylinder is L/4 and when they are placed in end to end, they just fill the hollow tube completely. To facilitate the counting of vibrations of the Maxwell's needle, a small plane mirror is attached to the specimen wire and observations are taken with the help of a telescope scale arrangement.

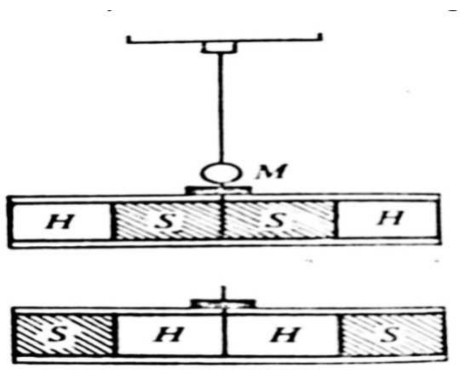


Fig.15.5(a) and (b)

Theory and procedure :

First, the solid cylinder S, S are placed in the inner position and the hollow cylinders H, H in the outer position as shown in fig. 15.5(a). The system is allowed to perform torsional oscillation. Let T_1 be the time period of vibration, then

$$T_1 = 2\pi\sqrt{\frac{I1}{c}} \qquad (1)$$

where I_1 is the moment of inertia of the suspended system about the wire as axis and C the torsional couple per unit twist of the wire.

Next, the position of hollow and solid cylinders are interchanged (fig. 15.5(b)). Now, if I_2 be the moment of inertia of the system about the wire as axis, then the time period is given by

$$T_2 = 2\pi\sqrt{\frac{I2}{c}} \qquad (2)$$

Solving the above equation, we get

$$C= 4\pi^2 \frac{I_2-I_1}{T_2^2-T_1^2}$$

$$\text{But, } C=\frac{\eta\pi r^4}{2l} \qquad (3)$$

where l and r are the length and radius of the specimen wire respectively,

$$\frac{\eta\pi r^4}{2l} = 4\pi^2 \frac{I_2-I_1}{T_2^2-T_1^2}$$

$$\eta = \frac{8\pi l\,(I_2-I_1)}{(T_2^2-T_1^2)r^4} \qquad (4)$$

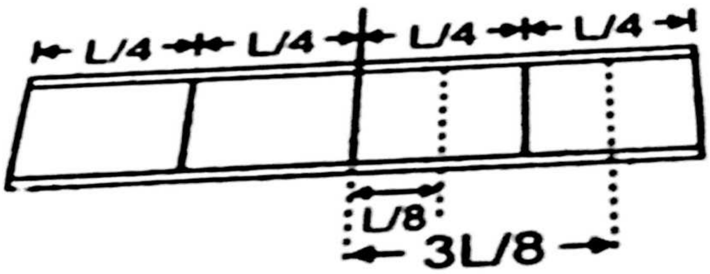


Fig.15.6

Determination of I_2-I_1 :

Let I_0 be the moment of inertia of the outer brass tube about a vertical axis passing through its centre of mass, and I_h and I_s , the moment of inertia of the hollow and solid cylinder$_2$ respectively about the vertical axis passing through their respective centre of mass. The centre of mass of the inner and outer cylinders are at distances L/8 and 3L/8 respectively from the axis of oscillation (fig. 15.6). Let m_h be the mass of each hollow cylinder and m_s that of each solid cylinder. Then, using the theorem of parallel axes, we get

$$I_1=I_0 + 2I_s + 2m_s\left(\frac{L}{8}\right)^2 + 2I_h + 2m_h\left(\frac{3L}{8}\right)^2$$

and $$I_2=I_0 + 2I_h + 2m_h\left(\frac{L}{8}\right)^2 + 2I_s + 2m_s\left(\frac{3L}{8}\right)^2$$

$$\therefore I_2 - I1 = 2m_s[(\frac{3L}{8})^2 - (\frac{L}{8})^2] + 2m_h[(\frac{L}{8})^2 - (\frac{3L}{8})^2]$$

$$= 2m_s[(\frac{3L}{8})^2 - (\frac{L}{8})^2] - 2m_h[(\frac{3L}{8})^2 - (\frac{L}{8})^2]$$

$$= 2(m_s - m_h)\,[(\frac{3L}{8})^2 - (\frac{L}{8})^2]$$

$$= 2(m_s - m_h)\,[\frac{(9L^2 - L^2)}{64}]$$

$$= (m_s - m_h)\,\frac{8L^2}{32}$$

$$= (m_s - m_h)\,\frac{L^2}{32}$$

The time period T_1 and T_2 are measured by lamp and scale arrangement, l and L are measured directly by scale, r is obtained by measuring the diameter of wire with the help of screw gauge. Thus η, the modulus of rigidity of the wire is calculated by substituting all the values in equation (6). Maxwell's needle is superior to torsional pendulum in two ways.

(i) In Maxwell's needle, the weight of the suspended system remains same in both parts of the experiment. Hence, the torsional constant C of the wire remains constant.

(ii) In this method, we need not find moment of inertia but find masses which can be known with greater precision.

Solved problems

Q1- A wire of length 1m and diameter 2mm is clamped at one of its ends. Calculate the couple required to twist the other end by 45˚. ($\eta = 5\times10^{10}$ N/m^2)

Sol- The couple required to twist the free end of a clamped wire of length l and radius r through an angle of Φ radian is

$$\tau = \frac{\pi\eta r^4 \Phi}{2l}$$

For Φ=45˚ = π/4 radian, we have

$$\tau = \frac{\pi^2 \eta r^4}{8l}$$

Here, l=1m, r=1.00mm= 10^{-3}m and η= 5×10^{10} N/m^2

$$\therefore \quad \tau = \frac{(3.14)2\, X\, (5\, X\, 10^{10})X\, (10^{-3})4}{8X1}$$

$$= 6.2\text{x}10^{-2} \text{ Nm}$$

Q2- Two solid cylinders of the same material having length l and 2l and radii r and 2r, are joined coaxially. Under a couple applied between the free ends, the shorter cylinder shows a twist of 30˚**.** Calculate the twist of the longer cylinder.

Sol- Let τ be the couple that produces a twist Φ in the shorter cylinder and twist Φ**'** in the larger cylinder. Then,

$$T = \frac{\pi\eta r^4 \Phi}{2l}$$

$$= \frac{\pi\eta(2r)^4\Phi'}{2(2l)} \pi\eta\ (2r)^4\ \Phi' / 2\ (2l)$$

$$\Phi' = \frac{\Phi}{8} = \frac{30^{\circ}}{8} = 3.75$$

15.4 Bending of beam:

A rod of uniform rectangular or circular cross section, whose length is very large in comparison to its thickness or radius is called a beam.

15.4.1 Longitudinal filament:

A rectangular beam may be supposed to be made up of a number of thin layers placed in contact and parallel to one another. Similarly, a cylindrical beam may be supposed to be made up of thin cylindrical layers placed in contact and coaxial to each other. Further, each layer may be considered as a collection of thin fibres lying parallel to the length of the beam. These fibres are called 'longitudinal filaments' of the beam.

15.4.2 Neutral surface:

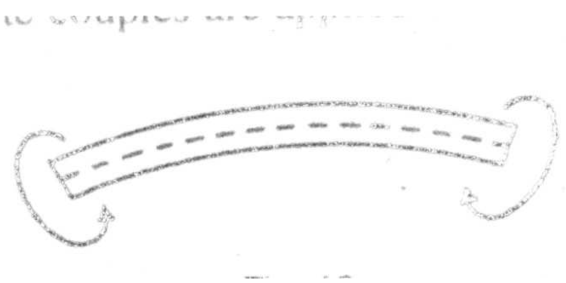


Fig.15.4.1

When equal and opposite couples are applied at the ends of a beam in a plane parallel to its length, the beam bends into a circular arc. Fig 15.4.1 shows the vertical sections of such a bent beam. In bending, the filaments on the convex side of the beam are extended in length, while those on the concave side are compressed.

There is, however, a plane in the beam in which the filaments remain unchanged in length. This is called the 'neutral plane' or 'neutral surface'. It passes through the centres of the areas of the cross-section of the beam. In fig. 15.4.1, the dotted lines represent the intersection of the neutral surface by the plane of the diagram.

15.4.3 Plane of bending:

The plane in which the beam bends is called the 'plane of bending'. It is the plane parallel to the long axis of symmetry of the beam and passing through it and the centre of curvature of the bent beam. In fig. 15.4.2, ABCD is the plane of bending.

15.4.4 Neutral axis:

The line along which the neutral surface of a beam intersects the plane of bending is called the 'neutral axis'.

15.4.5 Bending moment:

When a beam is bent by an external applied couple, an internal restoring couple is developed at each cross-section of the beam due to its elasticity. In the equilibrium state, the restoring couple is equal and opposite to the external couple. The magnitude of the restoring couple is called the 'bending moment' and is equal to the external couple.

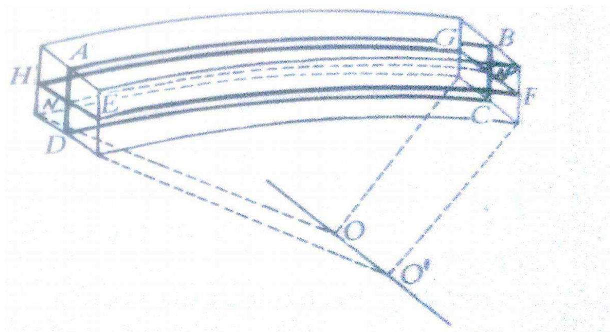


Fig.15.4.2

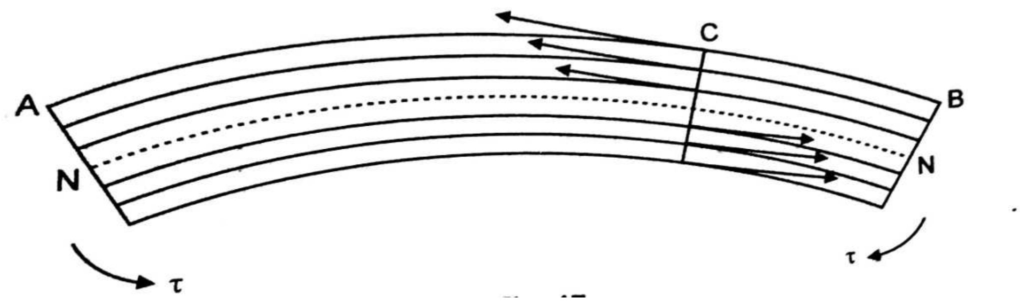


Fig15.4.3

Fig. 15.4.3 represents the vertical section of a beam AB bent under the action of two equal and opposite couples τ at each ends. The dotted arc NN represents the intersection of the neutral surface by the plane of the diagram. The other arcs represent the filaments in this plane

Let us divide the beam into two equal parts by a plane C and consider the equilibrium of the part CB. Due to bending, the filament of the beam above the neutral surface arc extend while those below are compressed. The change in length of any filament is proportional to its distance from the neutral surface, i.e. the longitudinal strains in the filament increases from zero at the neutral surface to a maximum at the upper and lower surfaces of the beam. Corresponding to these strains are stresses on the cross-section of these filaments. Above the neutral surface, the portion of a filament in the part AC exerts an external force on its portion in the part CB. Similarly, below the neutral surface, the portion of a filament in the part AC exerts a compressional force on its portion in the part CB. These extensional and compressional forces decrease as we go toward the neutral surface as indicated by the arrows. They can be paired up to form a number of anti-clockwise couples, whose resultant is the restoring couple acting at the section C. The magnitude of this couple is called the 'bending moment'. Obviously it is exerted by the part AC of the beam over the part CB. As the part CB is in equilibrium, the anticlock-wise couple at C must be equal to the clockwise external couple τ at B.

Now, we calculate the restoring couple. Fig. 15.4.4 represents the part CB of the bent beam and the forces acting over the section C. Let it subtend an angle Φ at the centre of curvature of the neutral surface NN'. Let us consider a filament PQ at a distance z above the neutral surface. From the fig., it follows that

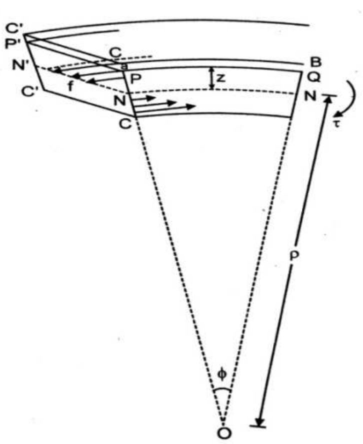


Fig.15.4.4

$$PQ = (R+z)\Phi$$

$$\text{and } NN' = R\Phi$$

Before bending, the length of PQ was same as that of NN', i.e. $R\Phi$. Now, the extension in the filament is

$$(R+z)\Phi = R\Phi = z\Phi$$

Hence the strain in the filament is

$$\text{Strain} = \frac{change\ in\ length}{original\ length}$$

$$= \frac{z\Phi}{R\Phi} = \frac{z}{R}$$

If 'f' be the force acting on the cross-section of the filament and 'a' the area of the cross-section, then the stress = f/a and so the value of young's modulus

$$Y = \frac{stress}{strain} = \frac{f/a}{z/R}$$

$$= \frac{fR}{za}$$

$$f = \frac{Yaz}{R}$$

The moment of this force about the neutral surface is

$$f.z = \frac{Ya}{R} z^2$$

The sum of the moments of the force acting over the whole cross-section C is the magnitude of the restoring couple or the bending moment.

Therefore, bending moment = Σfz

$$= \Sigma \frac{Ya}{R} z^2$$

$$= \frac{Y}{R} \Sigma a z^2$$

Now, Σ (az^2) is the geometrical moment of inertia (I) of the cross-section about the neutral surface. It is analogous to the moment of inertia with the difference that mass is replaced by area. Now, we have

$$\text{Bending moment} = \frac{YI}{R}$$

Since, at equilibrium the restoring couple is equal to the external couple τ, we have

$$\tau = \frac{YI}{R}$$

This is the relation between external couple τ and the radius R of the circular arc into which the couple bends the beam. This is the fundamental equation for the bending of the beam. The above equation shows that larger the value of Y, larger is the resistance to bend the beam of that material.

15.4.6 Flexural rigidity:

The quantity YI which is defined as the external bending moment required to produce unit radius of curvature is called 'flexural rigidity' of the beam to bend.

If the beam is of rectangular cross-section having breadth 'b' and depth 'd', then the area of cross-section is b.d and the square of radius of gyration ($k^2 = d^2/12$), then

$$I = \text{area x } k^2$$

$$= b.d \text{ x } \frac{d^2}{12}$$

$$= b.\frac{d^3}{12}$$

$$\text{Bending moment} = \frac{Yb.d^3}{12R}$$

Similarly, in cylindrical rod of radius r, the area of cross-section is πr^2 and $k^2 = r^2/4$,

Thus

$$I = \pi r^2 \text{ x } \frac{r^2}{4}$$

$$= \frac{\pi r^4}{4}$$

$$\text{Bending moment} = \frac{Y\pi r^4}{4R}$$

15.4.7 Stiffness of a beam:

The stiffness of a beam is taken to be the ratio between the maximum deflection of its loaded end and its span and is denoted by the symbol 1/n. for steel girders with a large span, 1000<n<2000, and for those with a smaller span, 500<n<700. For timber beams, n>=360.

15.5 The cantilever

A horizontal beam, fixed at one end, is called a Cantilever. Let us consider a thin, uniform and light beam of length l clamped horizontally at one end A(15.5.1). When it is loaded with a weight W at the free end B, the end B is depressed downward compared to A so that the beam undergoes bending. Such a system is called a Cantilever. Since the beam is light, the whole depression may be taken as due to the load W.

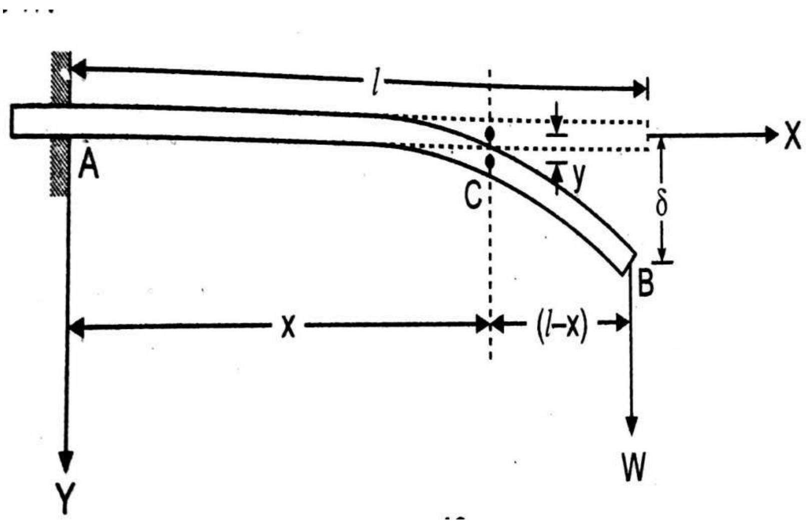


Fig.15.5.1

Let us take a section of the beam at C, distant x from A and consider the equilibrium of the part CB. Since the beam is fixed at A, the load W at B exerts an external torque tending to rotate the beam clockwise. Its magnitude is W(l-x). This torque is balanced by the anticlockwise restoring torque which is YI/R, as already derived in section 15.4.5. At equilibrium, therefore

$$W\,(l-x) = \frac{YI}{R} \qquad (1)$$

$$R = \frac{YI}{W(l-x)} \qquad (2)$$

This above equation shows that the radius of curvature R at a point of the beam is inversely proportional to (l-x), the distance of the point from the loaded end.

Now, let y be the depression at the point C. Taking the end A as origin, X and Y axes are drawn. Then, (x,y) are the co-ordinates of the point C, and the radius of curvature at this point is given by

$$R = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} \qquad (3)$$

where dy/dx is the slope of the tangent at the point (x,y). If the depression be within elastic limit, the slope will be small. Therefore, $(dy/dx)^2$ will be negligible compared to 1 and we can put

$$R = \frac{1}{d^2y/dx^2} \qquad (4)$$

Substituting R in eq.2, we get

$$\frac{1}{d^2y/dx^2} = \frac{YI}{W(l-x)}$$

$$\text{Or } \frac{d^2y}{dx^2} = \frac{W}{YI}(l\text{-}x)$$

On integration, we get

$$\frac{dy}{dx} = \frac{W}{YI}(lx - x^2/2) + A \qquad (5)$$

where A is constant of integration. At the clamped end A of the beam, the tangent is horizontal, i.e. at x=0, we have dy/dx=0. Substituting this in equation 5

$$A = 0$$

Thus, eq.5 becomes

$$\frac{dy}{dx} = \frac{W}{YI}(lx - x^2/2)$$

On further integration, we get

$$y = \frac{W}{YI}(lx^2/2 - x^3/6) + B$$

where B is again a constant of integration. Again at x=0, we have y=0 and so B=0

$$\therefore\ y = \frac{W}{YI}(lx^2/2 - x^3/6) \qquad (6)$$

This is the expression for the depression at a distance x from the fixed end.

At B, the free end (fig. 15.5.1) we have, x=l, and the depression y is maximum. Let it be equal to δ. Then, substituting l and δ for x and y respectively in eq.6, we get

$$\delta = \frac{W}{YI}(l^3/2 - l^3/6)$$

$$= Wl^3 / 3YI \qquad (7)$$

If the beam is rectangular in shape, $I = bd^3/12$, where b is breadth and d is the depth of the beam. Therefore, from eq.7, we get

$$\delta = 4Wl^3 / Ybd^3 \qquad (8)$$

If the beam is of circular cross-section of radius 'r', then $I = \pi r^4/4$. Then from eq.7, we get

$$\delta = 4Wl^3 / 3Y\pi r^4 \qquad (9)$$

These (eq.7 and eq.8) are the required expression for the depression.

The external load W applied at the free end of the cantilever exerts a bending torque at each section of the cantilever. Its value at a distance x from the fixed end is W(l-x), where l is the length of the cantilever. This is maximum for x=0, i.e. at the fixed end. Hence, the cantilever is more likely to break near the fixed end.

15.5.1 Transverse vibrations of a loaded cantilever:

Let us consider a cantilever of length l loaded at the free end with a mass M. Now let it be slightly depressed at its free end and then released. If y be the displacement of the free end of the cantilever at any time t, then the force F required to keep the cantilever with this displacement is given by (eq.7).

$$F = -3YIy / l^3$$

If d^2y/dt^2 be the acceleration produced in the mass M at that instant at the free end, then the force on the mass M is

$$M.\ d^2y/dt^2$$

Therefore, the equation of motion is

$$M\ d^2y / dt^2 = -3YIy / l^3$$

$$M\ d^2y / dt^2 + 3YIy / l^3 = 0$$

$$d^2y / dt^2 + 3YIy / M\ l^3 = 0$$

This is the differential equation of S.H.M., whose periodic time is given by

$$T = 2\pi \sqrt{\frac{Ml^3}{3YI}}$$

For a beam of rectangular cross-section of breadth 'b' and thickness 'd', $I=bd^3/12$.

Therefore,

$$T = \sqrt{\frac{M\ l^3}{Yb\ d^3}}$$

$$\text{Or}\ \ Y = 16\ \pi^2\ M\ l^3 / T^2\ bd^3$$

Using above equation, we can find Young's modulus of material of beam (cantilever). This method is specially suitable for the determination of Y of a wooden metre scale.

15.6 Shape of girders:

In the various cases of bending of beams discussed above, we have seen that the depression of the beam δ, is proportional to Wl^3/ YI. For a beam of rectangular area of cross-section, $I=bd^3 / 12$, so we have

$$\delta \; \alpha \; Wl^3 / Ybd^3$$

Thus for a given load W, the depression δ is directly proportional to l^3 and inversely proportional to b, d^3 and Y for its material.

For the depression δ to be small for a given load W, therefore, the length or span of the girder should be small and its breadth and depth large and also a large value of Y for the material.

Since in a supported or fixed beam, the middle portion gets depressed, its upper and lower halves get compressed and extended respectively. These compression and extension, and hence the corresponding stresses, are, as we know, the maximum at the upper and the lower surfaces progressively decrease to zero as we approach the neutral surface from either face. Obviously, therefore the upper and the lower surfaces of the beam must be stronger than the intervening part. That is why the two surfaces of a girder or iron nails (for railway tracks etc.) are made much broader than the rest of it, thus giving its cross-section the shape of the letter I.

This naturally affects a good deal of saving in the material of the girder, without appreciable impairing its strength.

Solved problems

Q1- A steel rod of length 50cm, width 2cm and thickness 1cm is bent into the form of an arc of radius of curvature 2.0m. Calculate the bending moment. Young's modulus of the material of the rod = 2×10^{11} N/m^2.

Sol- $b=2cm = 2x10^{-2}m$

$$d=1cm = 10^{-2}\ m$$

$$I = bd^3/12 = \frac{2*10^{-2}*10^{-6}}{12} = \frac{1}{6}\,10^{-8}$$

$\therefore$ Bending moment$= \frac{YI}{R} = \frac{2*10^{11}}{2} \text{x} \frac{1}{6}\,10^{-8} = 10^3/6 = 166.67$ Nm

Q2- Compare the loads required to produce equal depressions for two beams made of the same material and having the same length and weight with only difference that one has circular cross section while the other is square.

Sol- If l is the length of each bar, ρ its density, r is the radius of the circular bar and a each side of the face of the square bar, then

Mass of square bar= $a^2 l\rho$

Mass of circular bar= $\pi r^2 l\rho$

As, the mass of the bar are equal,

$\pi r^2 l\rho = a^2 l\rho$

$$\pi r^2 = a^2 \qquad \text{(i)}$$

Let I_1 be the geometrical M.I. of the square bar and I_2 that of the circular bar, then

Depression for square bar $y = \frac{W1}{YI1}\frac{l^3}{3}$ (ii)

And depression for circular bar $y = \frac{W2}{YI2}\frac{l^3}{3}$ (iii)

From (ii) and (iii), we get

$$= \frac{W1}{I1}\frac{W2}{I2}$$

$$\text{Or,} \quad \frac{W1}{W2} = \frac{11}{I2}$$

$$\text{Now, } I_1 = a^4/12$$

$$\text{and } I_2 = \pi r^4/4$$

$$\therefore \frac{W1}{W2} = \frac{a^4}{12} \text{X} \frac{4}{\pi r^4} = \frac{a^4}{3\pi r^4}$$

$$= \frac{\pi^2 r^4}{3\pi r^4} = \frac{\pi}{3} = 1.05$$

Q3- The end of a given strip cantilever depresses 10 mm under a certain load. Calculate the depression under the same load for another cantilever of same material 2 time in length, 2 times in width and 3 times in thickness.

Sol- The depression δ at the end of a cantilever is

$$\delta = Wl^3 / 3YI$$

For a rectangular cantilever $I = bd^3 / 12$

$$\therefore \delta = 4Wl^3 / Ybd^3$$

The depression for another cantilever of the same material (Y same) but

Length 2l, width 2b and thickness 3d would be

$$\delta' = 4W(2l)^3 / Y(2b)(3d)^3$$

$$= \frac{4}{27} 4Wl^3 / Ybd^3$$

$$= \frac{4}{27} \delta = \frac{4}{27} 10mm = 1.48mm$$

15.7 Summary:

When a cylinder is fixed at one end, and to the lower end a twisting couple is applied, then the resisting couple tending to oppose the twisting couple is calculated. This couple is greater for a hollow cylinder than for a solid one of same material, mass and length. The depression of a beam which is fixed at one end and loaded at the other is determined. It is also explained why the surface of the girder or iron rails for railway tracks are made much broader than the rest of it, giving the cross-section the shape of the letter I.

15.8 Model examination questions:

Short answer questions

1- Explain why a hollow cylinder is stronger than a solid cylinder of same length, mass and material?
2- Define the terms beam, neutral surface and neutral axis.
3- Differentiate between angle of twist and angle of shear.
4- What do you mean by torsional rigidity?
5- What is flexural rigidity?
6- Explain why steel girders and rails are made in the form of I-section.

Long answer questions:

1- What are torsional oscillations? Derive an expression for the twisting couple per unit angular twist for a hollow cylinder.
2- What is a cantilever? Obtain an expression for the depression at the free end of a thin light beam clamped horizontally at one end and loaded at the other.
3- What do you understand by the bending moment? Obtain an expression in the case of a uniform beam. Find also, the depression at any point of the beam.
4- Show that the bending moment for a thin uniform bar of rectangular cross-section is $Ybd^3/12R$.

Numerical questions:

1- A solid cylinder of radius 5cm is converted into a hollow cylinder of same mass and length and external radius 7cm. If the restoring couple per unit radian twist in original cylinder is c, deduce the same for the new hollow cylinder.

(Ans- c' = 2.92c)

2- A uniform rod of length 1m is clamped horizontally at one end. A weight of 0.1 kg is attached at the free end. Calculate the depression of the free end of the rod. The diameter of the rod is 0.02m. $Y=1*10^{10}$ N/m^2

(Ans- 4.1mm)

3- A cylinder of diameter 4cm and length 5cm is suspended horizontally by a steel wire of length 100cm and radius 0.02cm. Calculate the time of one oscillation. The coefficient of rigidity of steel is $8*10^{11}$ $dynes/cm^2$ and density of load is 11.4 gm/c.c.

(Ans- 6.586 secs)

4- A cantilever of length l and uniform cross-section shows a depression of 2cm at the loaded end. What will be the depressions at distances l/4, l/2 and 3l/4 from the fixed end?

(Ans- 0.34cm, 0.62cm, 1.27cm)

5- What couple must be applied to a wire 1m long and 2mm in diameter in order to twist one of its end through 45˚, when the other end remains fixed? ($\eta=5x10^{11}$ $dynes/cm^2$)

(Ans- 61.6 dyne cm)

15.9 Reference Books:

Prof D.S.Mathur	Mechanics	S.Chand publications
J.C. Upadhyay	Mechanics	Ram Prasad Publications
R.K. Shukla and Anchal Srivastava	Mechanics	New Age International Publications

D.N. Tripathi and R.B. Singh	Mechanics and Wave Motion	Kedarnath Ramnath Publications